

# Dynamics of purely meromorphic functions of bounded type

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## Abstract

We study the parameter plane of one-dimensional dynamically defined slices of meromorphic transcendental maps of finite type for which infinity is not an asymptotic value. More precisely we investigate *shell components* in these parameter planes which are components of parameter values for which a free asymptotic value is attracted to an attracting periodic orbit. Among other things, we prove that, in contrast to slices of parameter spaces for rational maps, where one sees Mandelbrot-like components that have a unique *center*, shell components do not have centers. Instead, they always have a unique *virtual center*, which may be at infinity, and corresponds to a map for which an iterate of the asymptotic value lands at a pole.

## 1 Introduction

Our starting point is a family  $\{f_\lambda : \mathbb{C} \rightarrow \widehat{\mathbb{C}}\}$  of transcendental meromorphic functions that depends holomorphically on a parameter  $\lambda = (\lambda_1, \lambda_2, \dots)$ . Iterating the function gives rise to a dynamical system. An interesting and generally very difficult problem is to understand what bifurcations occur in the dynamics as one varies the parameter. For simple families depending on a single complex parameter like the exponential family  $E_\lambda(z) = e^z + \lambda$ , and for related systems like  $\lambda \sin(z)$  and  $\lambda \tan(z)$ , this has been carried out more or less completely. (See for example, [RG03, KK97] as well as references quoted in this text). As a first step toward making the general problem tractable, in this paper we restrict ourselves to certain finite dimensional parameter spaces and define certain one-dimensional slices of these spaces chosen in a way that is “compatible” with the dynamics.

For rational maps, there is a substantial literature studying parameter spaces. The essential new feature for transcendental functions is that they have asymptotic values. These are, in effect, images of infinity along certain paths. They introduce new phenomena into the

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structure of the parameter spaces not seen for rational maps. The idea in this paper is to pin down the dynamics so as to allow only one “free” asymptotic value to vary. Our main results concern properties of the parameter regions for which this free asymptotic value is attracted to an attracting cycle. These regions, which we call “shell components”, are the analogues in this setting of the hyperbolic components for quadratic maps that appear, for example, as components of the Mandelbrot set. As we shall see, there are interesting similarities and dissimilarities between hyperbolic components of rational maps and our shell components.

To make these ideas precise, we consider the dynamical system formed by iterating meromorphic transcendental maps of the complex plane

$$f : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}.$$

Because of the essential singularity at infinity, the global degree is infinite and the dynamics is richer than for rational maps. Well known dynamical models, like the tangent family, are functions in this class. Newton’s method applied to entire functions, almost always yields a transcendental meromorphic map as the root finder function to be iterated.

The dynamical plane of such maps splits into two disjoint completely invariant subsets: the *Fatou set*, or points for which the family of iterates form a normal family in some neighborhood; and its complement, the *Julia set*. The Fatou set is open and contains, among other types of components, all basins of attraction of attracting periodic orbits. By contrast, the Julia set is closed, has no interior, and can be characterized as the closure of the repelling periodic points or, if the map is meromorphic, as the closure of the set of poles and prepoles. For general background on meromorphic dynamics we refer to the survey [Ber93] and the references therein.

A key role in the dynamics of a meromorphic map  $f$  is played by points in the set  $S(f)$ , the *singular values* of  $f$ ; these are either *critical values* (images of zeroes of  $f'$ , the *critical points*) or *asymptotic values* (points  $v = \lim_{t \rightarrow \infty} f(\gamma(t))$  where  $\gamma(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ). Indeed every periodic connected component of the Fatou set is, in some sense, associated to the orbit of a point in the closure of  $S(f)$ . For example, any basin of attraction must contain a singular value in its interior. Note that  $\overline{S(f)}$  coincides with the set of singularities of the inverse map.

Transcendental meromorphic functions with a finite number of singular values have finite dimensional parameter spaces [EL92, GK86]. These are known as *finite type* maps and denoted by

$$\mathcal{S} := \{f : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\} \text{ meromorphic} \mid \#S(f) < \infty\}.$$

Maps in class  $\mathcal{S}$  have several special properties such as the absence of wandering domains [Sul85, GK86, EL92, BKY92] or the existence of at most a finite number of attracting cycles.

Entire transcendental functions, whose dynamics have attracted a great deal of attention in the last few years, are special cases of transcendental meromorphic maps because they have no poles. This lack of poles forces the point at infinity to be, in some sense, a fixed point (with infinite multiplicity) and thus an asymptotic value. This has important consequences for the dynamics; some of these are shared by meromorphic maps with a finite number of poles (in which case infinity is also an asymptotic value). The simplest example is the exponential family  $E_\lambda(z) = e^z + \lambda$ . Meromorphic maps for which infinity is not an asymptotic value are, by contrast, at the other end of the spectrum; these necessarily have infinitely many poles.

The tangent family of maps  $T_\lambda(z) = \lambda \tan(z)$  [DK88, KK97, GK08] is the simplest example of these. We denote by

$$\mathcal{M}_\infty := \{f \in \mathcal{S} \mid \infty \text{ is not an asymptotic value}\},$$

the class of functions we concentrate on in this paper. Our main goal is to investigate some of the features of maps in  $\mathcal{M}_\infty$  and to see which of these are shared with entire transcendental functions in  $\mathcal{S}$ .

To center the discussion, we consider *dynamically natural slices* of these finite type maps. Roughly speaking, a dynamically natural slice is a one-parameter holomorphic family of meromorphic maps for which only one singular value is allowed to bifurcate at any given parameter value (see Definition 4.5 for details). We concentrate our attention on the behavior of a marked asymptotic value called the *free asymptotic value*, and in particular, on the connected components of parameters for which this marked point is attracted to an attracting cycle whose basin contains no other singular value. We call these components *shell components* and they are the natural generalization of the hyperbolic components one finds in parameter planes of rational maps. A dynamically natural slice is represented by a one-dimensional manifold isomorphic to the complex plane with a finite number of punctures (or *parameter singularities*). For normalization purposes, we require the parametrization to trace the position of the marked asymptotic value in an affine fashion. Most of the best known one-parameter slices such as quadratic polynomials, cubic polynomials with a super attracting fixed point, quadratic rational maps with a super attracting cycle, exponential or tangent functions, and so on, are dynamically natural slices of maps in different classes (see Sections 4.1 and 7). Most of our examples are drawn from transcendental families of meromorphic maps with a given number of asymptotic values and no critical points. These are families for which, like rational maps, an embedding into a complex manifold is known. In the Appendix we show how to extend these representations to somewhat more general functions.

Suppose  $\Lambda$  is a dynamically natural slice for the family  $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$  of meromorphic maps. Then,  $\Lambda$  contains different types of “distinguished” parameter values which are solutions of algebraic or transcendental equations. Examples are: *Misiurewicz parameters*, where a singular value lands on a repelling cycle; or *centers*, where a critical point (if it exists) is periodic. For meromorphic maps we find new distinguished parameters we call *virtual cycle parameters*. For these, some iterate of the asymptotic value is a pole. Their name is motivated by the existence of a *virtual cycle*, morally a “cycle” that contains the point at infinity and the asymptotic value (see Definition 6.7). Virtual cycle parameters are important in our discussion. They are quite abundant in the following sense (see Proposition 5.6).

**Proposition A.** *Suppose  $\Lambda$  is a dynamically natural slice for the family  $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$  of maps in  $\mathcal{M}_\infty$ . Let  $v(\lambda)$  denote the free asymptotic value. Then, Misiurewicz parameters and virtual cycle parameters are dense in the set*

$$\mathcal{B}(v) := \{\lambda_0 \in \Lambda \mid \{\lambda \mapsto f_\lambda^n(v_\lambda)\} \text{ is not a normal family in any neighborhood of } \lambda_0\}.$$

*As a consequence,  $\mathcal{B}(v)$  has no isolated points in  $\Lambda$ .*

The set  $\mathcal{B}(v)$  is known as the *activity locus* of the free asymptotic value [Gau12] and is part of the *bifurcation locus*  $\mathcal{B}$ , or the set of parameters for which at least one of the singular values

bifurcates. It follows from our definition of a dynamically natural slice, that the boundary of any shell component is a subset of  $\mathcal{B}(v)$ .

In many ways, a shell component  $\Omega$  is similar to the hyperbolic components we find in many dynamically natural slices. Indeed, a shell component can be parametrized by the multiplier map

$$\rho = \rho_\Omega : \Omega \longrightarrow \mathbb{D}^*$$

where  $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ , and  $\rho(\lambda)$  is the multiplier of the attracting cycle to which  $v(\lambda)$  is attracted. By definition, for functions in a shell component there are no critical points in the basin of the attracting cycle so the multiplier is never zero; therefore  $\Omega$  contains no center. It follows that  $\rho$  is a covering (see Theorem 6.4) and hence shell components are either simply connected and  $\rho$  has infinite degree or else they are isomorphic to  $\mathbb{D}^*$  where the puncture is a parameter singularity (see Corollary 6.5).

The parameters that play the role of centers in shell components are called “virtual centers” and they are in one-to-one correspondence with the virtual cycle parameters (see Theorem 6.10).

**Definition 1.1 (Virtual center).** Let  $\Omega$  be a shell component. We say that  $\lambda^* \in \partial\Omega \cap \Lambda$  is a *virtual center* of  $\Omega$ , if for every sequence of  $\lambda_n \in \Omega$  such that  $\lambda_n \rightarrow \lambda^*$  the multiplier  $\rho_\Omega(\lambda_n)$  tends to 0.

**Theorem B.** Suppose  $\Lambda$  is a dynamically natural slice for the family  $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$  of maps in  $\mathcal{M}_\infty$  and let  $\Omega$  be a shell component. Then  $\lambda^* \in \partial\Omega \cap \Lambda$  is a virtual center if and only if  $\lambda^*$  is a virtual cycle parameter.

As is the case for actual centers of hyperbolic components, every shell component has a unique virtual center.

**Theorem C.** Suppose  $\Lambda$  is a dynamically natural slice for the family  $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$  of maps in  $\mathcal{M}_\infty$  and let  $\Omega$  be a simply connected shell component. Let  $k \geq 1$  be the period of the attracting cycle throughout  $\Omega$ . Then:

- (a)  $\partial\Omega$  is locally connected.
- (b)  $\partial\Omega$  has a unique virtual center.
- (c) If  $k = 1$ , the virtual center is at infinity and therefore  $\Omega$  is unbounded.
- (d) Define the internal ray in  $\Omega$  of angle  $\theta$  as

$$R_\Omega(\theta) := \{\varphi(t + 2\pi i\theta), t \in (-\infty, 0)\}.$$

Then, all internal rays have one end at the virtual center and the other end at a point in  $\partial\Omega$  (which a priori could be infinity). If the virtual center  $\lambda^*$  is finite no internal ray has both ends at  $\lambda^*$ .

Theorem C is a summary of the corollaries of Theorem 6.13, namely 6.14, 6.15 and 6.17.

When the shell component is not simply connected then the puncture, which is a parameter singularity, plays the role of the virtual center.

Numerical experiments show that shell components of period larger than one are bounded, that is, they have their virtual center at a finite parameter value, so that it seems reasonable to conjecture that this is so in general. At present, we can only prove this fact for particular families like the tangent family.

The paper is structured as follows. In Section 2 we recall the definition of singular points of transcendental functions and we state some standard theorems on the covering and connectivity properties of their Fatou components. In Section 3 we discuss the properties of the dynamical planes of these functions. In Section 4 we define the *dynamically natural slices* of parameter spaces that are the main subject of the paper and in Section 4.1 we give a number of examples. In Section 5 we define various types of distinguished parameter values that lie in the bifurcation locus and prove Theorem A. Section 6 is the heart of the paper. We define *shell components* and prove the multiplier map is a holomorphic covering map. We prove some important technical propositions and conclude with proofs of Theorems B and C. In Section 7 we revisit the examples in Section 4.1 in the light of results of Section 6. We conclude with an Appendix where we discuss a theorem of Nevanlinna that allows us to characterize certain functions of finite type. We also extend this theorem to a somewhat larger class.

## 2 Preliminaries and Setup

### 2.1 Singularities of the inverse function

Let  $f$  be a transcendental entire or meromorphic map. A point  $v \in \mathbb{C}$  is a *singular value* of  $f$  if some branch of  $f^{-1}$  fails to be well defined in every small enough neighborhood of  $v$ . Singular values may be critical, asymptotic or accumulations thereof. If  $c$  is a critical point, that is, a zero of  $f'$  then its image  $v = f(c)$  is a *critical value*. If there is a path  $\gamma(t)$  such that  $\lim_{t \rightarrow 1} \gamma(t) = \infty$  and  $\lim_{t \rightarrow 1} f(\gamma(t)) = v$  then the limit  $v$  is an *asymptotic value* of  $f$ .

Singular values are classified in terms of the branches of  $f^{-1}$  as follows (see [BE95],[Ive06]):

**Proposition 2.1** (Classification of singularities). *Let  $f$  be an entire or meromorphic transcendental map. For any  $z \in \mathbb{C}$  and  $r > 0$ , let  $D(z, r)$  be the disk of radius  $r$  centered at  $z$ . Let  $U_r$  be a connected component of  $f^{-1}(D(z, r))$ , chosen such that  $U_r \subset U_{r'}$  if  $r < r'$ . Then there are only two possibilities:*

$$(a) \bigcap_r U_r = \{p\}, p \in \mathbb{C}$$

$$(b) \bigcap_r U_r = \emptyset.$$

*In case (a),  $f(p) = z$ , and either  $f'(p) \neq 0$  and  $z$  is a regular point, or  $f'(p) = 0$  and  $z$  is a critical value. In case (b), the chosen inverse branch with image  $U_r$  defines a transcendental singularity over  $z$ , and it can be shown that  $z$  is an asymptotic value for  $f$ .*

The map  $f$  is said to be of *finite type* if it has a finite number of singular values. Note that this implies all the singular values are isolated.

If an asymptotic value is isolated, the radius  $r$  in the above proposition can be chosen small enough so that  $f : U_r \rightarrow D(v, r) \setminus \{v\}$  is universal covering map. In this case  $U_r$  is

called an *asymptotic tract* for the asymptotic value  $v$  and  $v$  is called a *logarithmic singular value*. The number of distinct asymptotic tracts of a given asymptotic value is called its *multiplicity*.

In this paper, whenever we talk about the number of critical values and/or asymptotic values, we tacitly assume that we count with multiplicity.

An entire function  $f$  always has an asymptotic value at infinity.

**Proposition 2.2.** *A transcendental function  $f$  of finite type with finitely many critical points has at least two (logarithmic) asymptotic values, counted with multiplicity.*

*Proof.* Suppose such an  $f$  had only one asymptotic value  $v$ . Then  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}} \setminus \{v\}$  would be a covering map branched over its finitely many critical values. Since there are only finitely many critical points,  $f$  must be a finite degree branched cover of the plane and hence conformally equivalent to a polynomial. If  $f$  had no asymptotic values at all the map would be finite degree and hence rational.  $\square$

## 2.2 Mapping properties of meromorphic maps

In this paper we deal with the class  $\mathcal{S}$  of transcendental functions of *finite type*. Notice that because  $\mathcal{S}$  is not closed under composition, it is often useful to work with a larger class which is. We consider the class of functions which are meromorphic outside of a small set; more precisely

$$\widehat{\mathcal{S}} = \{f : \mathbb{C} \setminus E(f) \rightarrow \widehat{\mathbb{C}} \mid f \text{ is nonconstant meromorphic}\} \supset \mathcal{S}$$

where  $E(f) \subseteq \widehat{\mathbb{C}}$  a closed countable set of essential singularities.

Functions in the class  $\widehat{\mathcal{S}}$  were originally studied by M. E. Herring [Her98] and A. Bolsch [Bol99] who extended many properties known for rational, entire or meromorphic maps to maps of this class. The following statements are relevant for our setting. Given  $U \in \widehat{\mathbb{C}}$  we denote by  $\chi(U)$  its Euler characteristic (i.e.  $\chi(\widehat{\mathbb{C}}) = -2$  and  $\chi(U) = c(U) - 2$  when  $U$  is of connectivity  $c(U) \in \mathbb{N} \cup \{\infty\}$ ).

**Theorem 2.3** ([Hei57, Thm.4], [Bol99, Thm.1,3]). *Let  $f \in \widehat{\mathcal{S}}$ . Let  $V \subset \widehat{\mathbb{C}}$  be an arbitrary domain and let  $U$  be a connected component of  $f^{-1}(V)$  containing  $k$  critical points counted with multiplicity. Then either*

- (a) *there exists an integer  $n \in \mathbb{N}$ , with  $k \leq 2n - 2$ , such that every value of  $V$  is assumed exactly  $n$  times in  $U$  and*

$$\chi(U) = n\chi(V) + k;$$

*or*

- (b) *with at most two exceptions, every value of  $V$  is assumed infinitely often in  $U$ . In this case, if  $\chi(V) > 0$  then  $\chi(U) = +\infty$*

Observe that the first case occurs if and only if  $f|_U : U \rightarrow V$  is a proper map.

The following characterization was given in [BKL91] for invariant Fatou components of maps in  $\mathcal{S}$ . The same arguments applied to maps in  $\widehat{\mathcal{S}}$  extend it to Fatou components of any period.

**Theorem 2.4** ([Bol99, Cor.2 and Remark following it]). *If  $f \in \widehat{\mathcal{S}}$ , a periodic Fatou component of arbitrary period has connectivity 1, 2 or infinity. If it is doubly connected then it is a Herman ring or it is a doubly connected invariant component of infinite degree. In this case  $f$  must have infinitely many singular points and so is not in the class  $\mathcal{S}$ .*

The following classical and well known theorem on the classification of holomorphic coverings of the punctured disk  $\mathbb{D}^* = \{z \mid 0 < |z| < 1\}$  will be useful throughout the paper. The proof can be found, for example, in [Zhe10, Thm. 6.1.1.].

**Theorem 2.5.** *Let  $X$  be a Riemann surface and let  $h : X \rightarrow \mathbb{D}^*$  be an unbranched holomorphic covering map. Then one of the following statements holds.*

- (a) *There exists a conformal mapping  $\psi$  of  $X$  onto the left half plane  $\mathbb{H} = \{z : \operatorname{Re} z < 0\}$  such that  $f = \exp \circ \psi$ .*
- (b) *There exists a conformal mapping  $\psi$  of  $X$  onto  $\mathbb{D}^*$  such that for some  $n \in \mathbb{N}$ ,  $f = (\psi)^n$ .*

### 3 Dynamical plane

In this section we describe some properties of the dynamical plane for either entire or meromorphic finite type maps. In particular, we are interested in basins of attraction of attracting cycles containing a unique asymptotic value  $v$ . Note that because there are only finitely many singularities,  $v$  is isolated and is thus a logarithmic singularity. It follows that if  $V^*$  is a punctured neighborhood of  $v$  it has at least one asymptotic tract among the components of  $f^{-1}(V^*)$ .

We start with a simple observation about the connectivity of attracting basins.

**Lemma 3.1.** *Let  $f$  be a function of finite type and let  $\mathcal{A}$  denote the immediate basin of attraction of an attracting cycle of period  $p$ . Then either every component of  $\mathcal{A}$  is simply connected or they are all infinitely connected.*

*Proof.* Theorem 2.4 establishes that the connectivity of each component is 1, 2 or  $\infty$ . Since  $f$  is of finite type, and since the cycle is attracting and so not a cycle of Herman rings, it follows that the connectivity is either 1 or  $\infty$ . Now by Theorem 2.3, the connectivity of a component cannot be strictly larger than that of its preimage. Therefore, since the components form a cycle, either they are all simply connected or they are all infinitely connected.  $\square$

The next two lemmas describe dynamical properties of the subclass of meromorphic functions of finite type for which infinity is not an asymptotic value, namely  $\mathcal{M}_\infty$ . Recall from the general theory that the immediate attracting basin of a periodic cycle must contain a singular value. In the next lemma we describe properties of this basin when the singular value is an asymptotic value.

**Lemma 3.2.** *Let  $f \in \mathcal{M}_\infty$  and  $p \geq 2$ . Suppose  $A_0, \dots, A_{p-1}$  are the components of the immediate basin of attraction  $\mathcal{A}$  of an attracting  $p$ -cycle, indexed so that  $A_{i+1} = f(A_i)$  for all  $i$  (with indices taken mod  $p$ ). Assume also that the basin contains only one singular value  $v$ , which is an asymptotic value and that  $v$  belongs, say, to  $A_1$ . Then,*

- (a)  $A_0$  is unbounded and maps infinite to one onto  $A_1 \setminus \{v\}$ . Moreover infinity is accessible from  $A_0$  and  $\partial A_{p-1}$  contains a pole.
- (b) If the components of  $\mathcal{A}$  are simply connected then  $f : A_j \rightarrow A_{j+1}$  is one to one for all  $j \neq 0 \bmod p$ .

*Proof.* Since the asymptotic value  $v \in A_1$  is the only singular value in the basin, it follows that  $A_0$  must contain an asymptotic tract; this implies  $f : A_0 \rightarrow A_1 \setminus \{v\}$  is an infinite degree covering and hence that  $A_0$  is unbounded. If  $\delta(t)$  is a path in  $A_1$  such that  $\lim_{t \rightarrow 1} \delta(t) = v$ , then the component  $\gamma(t)$  of  $f^{-1}(\delta(t))$  in  $A_0$  is an asymptotic path; that is,  $\lim_{t \rightarrow 1} \gamma(t) = \infty$ , which proves  $\infty$  is accessible from  $A_0$ .

Let  $\gamma : [0, 1) \rightarrow A_0$  be a curve landing at infinity when  $t \rightarrow 1$  and consider the connected component  $\sigma$  of  $f^{-1}(\gamma)$  contained in  $A_{p-1}$ . By continuity, any accumulation point of  $\sigma(t)$  when  $t \rightarrow 1$  must be infinity or a pole. Since the set of poles is discrete, it follows that  $\sigma$  must actually land at infinity or at a pole of  $f$ . It cannot land on infinity because then infinity would be an asymptotic value. Therefore  $\sigma$  lands at a pole which must then be in  $\partial A_{p-1}$ . It now follows that any connected component of  $f^{-i}(\gamma)$  contained in  $A_{p-i}$  lands at a prepole of order  $i$  on  $\partial A_{p-i}$ , for all  $i \neq 0 \bmod p$ . This proves (a).

To see (b) assume that the basin is simply connected. From the Riemann-Hurwitz formula (see Theorem 2.3 (a)) we conclude that for all  $i = 1, \dots, p-1$ , the restrictions  $f|_{A_i}$  must be univalent.  $\square$

The next lemma characterizes the mapping properties near infinity of functions in  $\mathcal{M}_\infty$ .

**Lemma 3.3.** *Let  $f \in \mathcal{M}_\infty$ . Then there exists a neighborhood  $U$  of infinity and an infinite sequence of bounded disjoint disks  $\widehat{B}_i$ ,  $i \in \mathbb{Z}$ , such that*

$$f^{-1}(U) = \cup_i \widehat{B}_i.$$

*Each  $\widehat{B}_i$  contains a pole  $p_i$  and the restriction of  $f$  to  $\widehat{B}_i$  is a branched covering map whose order is the same as the order of the pole.*

*Proof.* Since infinity is not an asymptotic value, all the singular values are finite and we can find a closed disk  $D$  containing all of them. Set  $U = \widehat{\mathbb{C}} \setminus D$ ,  $U_\infty = U \setminus \{\infty\} \sim \mathbb{D}^*$ . Then if  $B_i$  is a component of  $f^{-1}(U_\infty)$ , the restriction  $f_i$  of  $f$  to  $B_i$  is a covering map

$$f_i : B_i \rightarrow U_\infty.$$

Because infinity is not an asymptotic value, by invoking Proposition 2.1 and enlarging  $D$  if necessary, we deduce that each  $B_i$  is homeomorphic to a punctured disk, and moreover, that these disks are all disjoint. The restricted maps  $f_i$  extend to the puncture by continuity and send it to infinity; thus the puncture is a pole  $p_i$ . Let  $\widehat{B}_i = B_i \cup \{p_i\}$ . The extended maps are conjugate to a map of the form  $\zeta \mapsto \zeta^k$  where  $k$  is the order of the pole. Thus if  $p_i$  is a simple pole, there is a single valued branch of the inverse,  $g_i = f_i^{-1}$  that maps  $U$  onto  $\widehat{B}_i$ . If  $k > 1$  then there are  $k$  such inverse branches. The disks  $\widehat{B}_i$  accumulate at infinity.  $\square$



## 4 Dynamically natural slices of parameter space

Spaces of transcendental functions with  $N < \infty$  singular values form finite dimensional spaces of dimension  $N$ . Loosely speaking, each singular value controls a different aspect of the dynamics of the function. In this paper we consider two different subclasses: entire functions which necessarily have an asymptotic value at infinity and meromorphic maps for which infinity is not an asymptotic value. We define one dimensional slices of these spaces characterized by dynamical properties.

The theory of holomorphic motions is a good tool for studying these parameter spaces. This theory for rational maps is given in [McM94] and it is adapted to meromorphic functions in [KK97]. We state the results from the latter reference that we use here.

**Definition 4.1** (Holomorphic family). A *holomorphic family* of meromorphic maps over a complex manifold  $X$  is a map  $\mathcal{F} : X \times \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ , such that  $\mathcal{F}(x, z) =: f_x(z)$  is meromorphic for all  $x \in X$  and  $x \mapsto f_x(z)$  is holomorphic for all  $z \in \mathbb{C}$ .

**Definition 4.2** (Holomorphic motion). A *holomorphic motion* of a set  $V \subset \widehat{\mathbb{C}}$  over a connected complex manifold with basepoint  $(X, x_0)$  is a map  $\phi : X \times V \rightarrow \widehat{\mathbb{C}}$  given by  $(x, v) \mapsto \phi_x(v)$  such that

1. for each  $v \in V$ ,  $\phi_x(v)$  is holomorphic in  $x$ ,
2. for each  $x \in X$ ,  $\phi_x(v)$  is an injective function of  $v \in V$ , and,
3. at  $x_0$ ,  $\phi_{x_0} \equiv \text{Id}$ .

A holomorphic motion of a set  $V$  respects the dynamics of the holomorphic family  $\mathcal{F}$  if  $\phi_x(f_{x_0}(v)) = f_x(\phi_{x_0}(v))$  whenever both  $v$  and  $f_{x_0}(v)$  belong to  $V$ .

The following equivalencies are proved for rational maps in [McM94] and extended to the transcendental setting in [KK97].

**Theorem 4.3.** *Let  $\mathcal{F}$  be a holomorphic family of meromorphic maps with finitely many singularities, over a complex manifold  $X$ , with base point  $x_0$ . Then the following are equivalent.*

- (a) *The number of attracting cycles of  $f_x$  is locally constant in a neighborhood of  $x_0$ .*
- (b) *There is a holomorphic motion of the Julia set of  $f_{x_0}$  over a neighborhood of  $x_0$  which respects the dynamics of  $\mathcal{F}$ .*
- (c) *If in addition, for  $i = 1, \dots, N$ ,  $s_i(x)$  is are holomorphic maps parameterizing the singular values of  $f_x$ , then the functions  $x \mapsto f_x^n(s_i(x))$  form a normal family on a neighborhood of  $x_0$ .*

**Definition 4.4** ( $J$ -stability). A parameter  $x_0 \in X$  is a  $J$ -stable parameter for the family  $\mathcal{F}$  if it satisfies any of the above conditions. We denote by  $X^{stab}$ , the set of  $J$ -stable parameters for the family  $\mathcal{F}$ .

The set of non  $J$ -stable parameters is precisely the set where bifurcations occur, and it is often called the *bifurcation locus* of the family  $\mathcal{F}$ , and denoted by  $\mathcal{B}_X$ . In families of maps with more than one singular value, however, it makes sense to consider subsets of the bifurcation locus where only some of the singular values are bifurcating, in the sense that the families  $\{g_n^i(x) := f_x^n(s_i(x))\}$  are normal in a neighborhood of  $x_0$  for some values of  $i$ , but not for all. We define

$$\mathcal{B}_X^i := \mathcal{B}_X(s_i) = \{x_0 \in X \mid \{g_n^i(x)\} \text{ is not normal in any neighborhood of } x_0 \text{ in } X.\}$$

This is also known as the *activity locus (in  $X$ )* of the singular value  $s_i$  (see [Gau12]).

In this paper we investigate one dimensional slices of holomorphic families in which the activity loci of the different singular values are disjoint. This means that at any given parameter on our slice only one of the singular values is allowed to bifurcate. We will require our slices to be dynamically defined in a sense we describe below. Such slices can be found, for example, by fixing the dynamical behavior of all but one of the singular values.

**Definition 4.5 (Dynamically natural slice).** Let  $\mathcal{F}$  be a holomorphic family of entire or meromorphic maps over  $X$  (and by abuse of notation also denote the set of maps  $\{f_x\}_{x \in X}$  by  $\mathcal{F}$ ). Assume  $S(f_x)$ , the set of finite singular values of  $f_x$ , counted with multiplicity has cardinality  $N < \infty$  for all  $x \in X$ . A one dimensional subset  $\Lambda \subset X$  is a *dynamically natural slice* with respect to  $\mathcal{F}$  if the following conditions are satisfied.

- (a)  $\Lambda$  is holomorphically isomorphic to the complex plane with finitely many points removed. The removed points are called *parameter singularities*. By abuse of notation we denote its image in  $\mathbb{C}$  by  $\Lambda$  again, and denote the variable in  $\mathbb{C}$  by  $\lambda$ .
- (b) The singular values are given by distinct holomorphic functions  $s_i(\lambda)$ ,  $i = 1, \dots, N-1$ , and an asymptotic value  $v_\lambda$  that is an affine function of  $\lambda$ . We call  $v_\lambda$  the *free asymptotic value*; we require that  $\mathcal{B}_\Lambda(v_\lambda) \neq \emptyset, \Lambda$ .
- (c) The poles (if any) are given by distinct holomorphic functions  $p_i(\lambda)$ ,  $\lambda \in \Lambda$ ,  $i \in \mathbb{Z}$ .
- (d) There are  $N-1$  distinct attracting cycles whose period and multiplier are constant for all  $\lambda \in \Lambda$ .
- (e) Suppose  $v_{\lambda_0}$  is the only singular value in  $\mathcal{A}_{\lambda_0}$ , the basin of attraction of an attracting cycle whose multiplier is not a constant function of  $\lambda$ . Then the slice  $\Lambda$  contains, up to affine conjugacy, all meromorphic maps  $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  that are quasiconformally conjugate to  $f_{\lambda_0}$  in  $\mathbb{C}$  and conformally conjugate to  $f_{\lambda_0}$  on  $\mathbb{C} \setminus \mathcal{A}_{\lambda_0}$ .
- (f)  $\Lambda$  is maximal in the sense that if  $\Lambda' = \Lambda \cup \{\lambda_0\}$  where  $\lambda_0$  is a parameter singularity, then  $\Lambda'$  does not satisfy at least one of the conditions above.

From now on, as long as it is understood from the context, we will drop the subindex  $\Lambda$ . In other words  $\mathcal{B} := \mathcal{B}_\Lambda$ .

*Remark 4.6.* Condition (d) could actually be weakened for our purposes. Morally what we really need is that  $N-1$  dynamical phenomena are kept constant throughout the slice

so that only one singular value is free at any given parameter value. For example, if two singular orbits were eventually equal (i.e. shared the same “tail”) throughout the slice, that would count as one dynamical phenomenon. Another possibility is that throughout the slice some singular value lands on an attracting cycle after a fixed number of iterations. A third possibility is that there is an asymptotic value whose orbit is tied to the orbit of the free asymptotic value by a relation which persists throughout the slice. An example of this last possibility is the tangent map, for which the two singular orbits are symmetric. Although most of our results hold in these more general situations, for simplicity of exposition we choose to require condition (d) as stated. We will, however, also consider dynamical slices in the more relaxed sense at some points and some examples in the paper.

*Remark 4.7.* Condition (e) is generally easy to verify in concrete families. If the initial holomorphic family is in some sense complete, and the slice is determined by the required dynamical conditions, deformations can be controlled. This will become clearer in the examples in Section 4.1. If, however, we relax condition (d) as in the above remark, condition (e) may have to be modified accordingly. For example, if there is a relation tying the orbit of a non-free asymptotic value to that of the free asymptotic value, condition (e) would say that up to affine conjugacy, all meromorphic maps  $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  that are quasiconformally conjugate to  $f_{\lambda_0}$  in  $\mathbb{C}$ , conformally conjugate to  $f_{\lambda_0}$  on  $\mathbb{C} \setminus \mathcal{A}_{\lambda_0}$  and whose asymptotic values satisfy the given relation must lie in the slice. The tangent family is an example.

*Remark 4.8.* Condition (f) is imposed to avoid artificial parameter singularities. In general, these singularities occur because the functions become constant or the number of singular values drops, or some poles escape to infinity when approaching these values.

*Remark 4.9.* Suppose  $\Lambda$  is a dynamically natural slice. For a given parameter value  $\lambda_0 \in \Lambda$ , if the free asymptotic value  $v_{\lambda_0}$  is attracted to an attracting cycle, it follows that the function  $f_{\lambda_0}$  is hyperbolic and therefore  $\lambda_0 \in X^{stab}$ . Because there are  $N - 1$  attracting basins which need to attract  $N - 1$  different singular values, only one of the singular values can be active at any given parameter. That is, if we set  $\mathcal{B}^0 := \mathcal{B}(v_{\lambda})$ , then for  $i, j = 0, 1, \dots, N - 1$ ,

$$\mathcal{B}^i \cap \mathcal{B}^j = \emptyset$$

whenever  $i \neq j$ . If we relax the definition of a dynamically natural slice as indicated in Remark 4.6 above, then we must also allow the possibility that  $\mathcal{B}^i = \mathcal{B}^j$  for some  $i \neq j$ . This is the case, for example, in the tangent family.

## 4.1 Examples

To find examples we start with some fairly simple entire or meromorphic functions, as described in the appendix, and pre-compose or post-compose with rational functions. The coefficients of these rational functions determine a natural parameter space. To define a dynamically natural family, we must cut down the dimension and show that condition (e) holds.

**Example 1.** The family of meromorphic functions with exactly two asymptotic values and no critical points is given by

$$\mathcal{F}_2 = \left\{ \frac{ae^z + be^{-z}}{ce^z + de^{-z}}, \ a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}.$$

See the Appendix and the references there. The asymptotic values are  $a/c$  and  $b/d$ . If  $b$  or  $c$  is equal to zero, the function is the exponential function and infinity is an asymptotic value. Otherwise both asymptotic values are finite. This is really a two dimensional family because we can normalize so that  $ad - bc = 1$  and conjugate by an appropriate affine transformation to reduce to two parameters. We will look at several different dynamically natural slices of  $\mathcal{F}_2$  and use different affine transformations for each.

- (a) The exponential family  $E_\lambda(z) = e^z + \lambda$  is one of the best understood dynamically natural slices of  $\mathcal{F}_2$ . It contains, up to affine conjugacy, every entire transcendental map with one finite asymptotic value and no critical values; property (d) holds. Since  $E_\lambda$  is entire, one of the asymptotic values is at infinity. The finite asymptotic value is  $v_\lambda = \lambda$ . In this case  $\Lambda = \mathbb{C}$ . Note that another normalization for this family in common use is one that fixes the second asymptotic value at the origin. The family then takes the form  $ae^z$  and the parameter is an affine function of the first iterate of the finite asymptotic value. Standard references include [BR75, DFJ02, Sch03, RG03].
- (b) Another dynamical slice that can be extracted from  $\mathcal{F}_2$  is formed by requiring the origin to be a fixed point with persistent multiplier  $\rho_0$ ,  $|\rho_0| < 1$ . The resulting slice is  $\Lambda = \mathbb{C} \setminus \{0, \rho_0/2\}$  and the maps have the form

$$f_\lambda(z) = \frac{e^z - e^{-z}}{(1/\lambda)e^z + (1/\mu)e^{-z}},$$

with  $\mu = \rho_0\lambda/(2\lambda - \rho_0)$ . The asymptotic values are  $\lambda$  and  $-\mu$  and at least one of them is attracted by the origin. See [GK08] for a discussion of properties of maps in the slice where  $\rho_0 = 1/2$ . To see that condition (e) is satisfied, observe that any conjugacy which is conformal in the basin of attraction of 0 keeps the multiplier of this fixed point unchanged. Since the resulting conjugate map again has two asymptotic values and no critical points, it belongs to  $\mathcal{F}_2$ . Using an affine conjugacy, we may require that the new fixed point be the origin again; thus the new map belongs to this family.

- (c) A slice of  $\mathcal{F}_2$  in the relaxed sense is given by the tangent family of maps

$$T_\lambda(z) = \lambda \tan z$$

with  $\Lambda = \mathbb{C}^*$  and  $v_\lambda = \lambda i$  [DK88, KK97, KY06]. This is a dynamically natural slice in the relaxed sense of Remark 4.6 because the second asymptotic value is  $s_1(\lambda) = -v_\lambda = -\lambda i$  and  $T_\lambda$  is symmetric. It follows that the bifurcation locus is the same for both singular values. Condition (e) here means that the relation  $s_1(\lambda) = -v_\lambda = -\lambda i$  must persist under deformation.

- (d) To form another slice in the relaxed sense, we again keep the origin fixed and fix one of the asymptotic values at  $\pi i$  so that its image is 0. The other asymptotic value is  $\lambda$ . Since the dynamical behavior of one asymptotic value is pre-periodic, but the multiplier at the origin is a function of  $\lambda$ , this is a dynamically natural slice in the relaxed sense. The functions have the form

$$f_\lambda(z) = \frac{e^z - e^{-z}}{(1/\lambda)e^z + (i/\pi)e^{-z}}.$$

We take the slice defined by the parameter  $v_\lambda = \lambda$ . The parameter singularities are at 0 and  $\pi i$  where the denominator becomes 0 or infinity, or the asymptotic values colide.

**Example 2.** Functions in  $\mathcal{F}_2$  can be composed with rational or polynomial functions and, up to affine conjugation, the dynamics of the composed functions remain invariant under quasiconformal deformations supported on the Fatou set (see Theorem 8.3 in the Appendix). Below we describe dynamically natural slices formed by pre and post composing with a quadratic polynomial  $Q(z) = az^2 + bz + c$ .

- (a) If we start with an  $f \in \mathcal{F}_2$  with finite asymptotic values and pre-compose by a degree two polynomial,  $Q(z) = az^2 + bz + c$ , the resulting function

$$g(z) = f(az^2 + bz + c)$$

has the same two asymptotic values as  $f$  does. Since each asymptotic tract of  $f$  has two pre-images under  $Q$ , each asymptotic value of  $g$  has two asymptotic tracts and so has multiplicity two. Since  $f$  has no critical points,  $g$  has a single critical point at  $-b/2a$  and a single critical value at  $f(-b/2a)$ . If we assume the origin is fixed, then  $c = 0$ . If we assume it is also a critical value then  $b = 0$ . Now making one of the asymptotic values land on the fixed point 0, we can form a dynamically natural slice. This determines the coefficient  $a$ . This is a slice in the strict sense because the origin is a super-attractive fixed point. The functions in the slice can then be written

$$g_\lambda = \frac{e^{z^2} - e^{-z^2}}{\frac{1}{\lambda}e^{z^2} + (\frac{1}{\sqrt{\pi i}})e^{-z^2}}$$

The asymptotic values are  $\lambda$  and  $-\sqrt{\pi i}$ . The singular parameter values are at 0 and  $-\sqrt{\pi i}$ . Theorem 8.3 implies that condition (d) holds.

- (b) If we start with an  $f \in \mathcal{F}_2$  with finite asymptotic values and post-compose by a degree two polynomial,  $Q(z) = az^2 + bz + c$ , the resulting function

$$h = af^2 + bf + c,$$

has two asymptotic values at the images of the asymptotic values of  $f$  under  $Q$ . Since  $f' \neq 0$ , the critical points are the infinite set of points,  $f^{-1}(-b/2a)$ , and there is a unique critical value at  $-b^2/(4a) + c$ . As in the above example, we can make a dynamically natural slice by making the origin a superattractive fixed point and fixing one of the asymptotic values at a preimage of the origin. We then have

$$h_\lambda(z) = \left( \frac{e^z - e^{-z}}{(\frac{1}{\sqrt{\lambda}})e^z + (\frac{1}{\sqrt{\pi i}})e^{-z}} \right)^2.$$

The asymptotic values are at  $\lambda$  and  $\pi i$  and the singular parameter values are at 0 and  $-\sqrt{\pi i}$ . We can apply Theorem 8.3 again to see that condition (d) holds. Notice that this function is semiconjugate to that in (a).

- (c) We can form a dynamically natural slice, in the relaxed sense, to the functions  $h = af^2 + bf + c$ , with  $f \in \mathcal{F}_2$  by assuming the origin is a super attracting fixed point and that the two asymptotic values coincide. In this case the functions in the slice each have one asymptotic value of multiplicity two. We can write these functions as

$$\lambda \tanh^2 z.$$

The parameter  $\lambda$  is the free (double) asymptotic value and so determines a dynamically natural slice of the parameter space  $\Lambda = \mathbb{C} \setminus \{0\}$ . Condition (e) is satisfied in the modified sense.

**Example 3.** Our last example is another family of functions with two singular values; these are entire functions with one critical value and one finite asymptotic value. Any entire function of finite order with two singular values, one of which is a fixed simple critical point, and the other a finite asymptotic value with one finite preimage, is affine conjugate to

$$D_a(z) = a(e^z(z+1) + 1), \quad a \in \mathbb{C}^*$$

for some  $a \in \mathbb{C}^*$ . (See Theorem 8.8 in the appendix.) The affine conjugacy is used to make the parameter unique by putting the critical point at 0 and the asymptotic value at  $a$  so that it is the free asymptotic value. Thus the maps  $D_a$ , with  $\Lambda = \mathbb{C}^*$ , form a dynamically natural slice.

## 5 Distinguished parameter values

In this section we discuss special parameter values for which the forward orbit of some singular value is finite.

**Definition 5.1 (Misiurewicz parameter).** A parameter  $\lambda$  (or the map  $f_\lambda$ ) is called *Misiurewicz* if there is a singular value  $s \in S(f)$  such that  $f_\lambda^n(s)$  is a repelling periodic point for some  $n \geq 0$ .

*Remark.* It might happen that for  $\lambda$  in a dynamically natural slice, some iterate of the free asymptotic value  $v_\lambda$  lands on an attracting or neutral periodic cycle  $\mathbf{a}$ . If  $\mathbf{a}$  is attracting or parabolic, there must be a singular value  $v' \neq v_\lambda$  of  $f_\lambda$  in the immediate basin of attraction of  $\mathbf{a}$ . Because we are in a dynamically natural slice  $\mathbf{a}$  must be one of the  $N - 1$  persistent attracting cycles. Since  $v_\lambda$  belongs to a preperiodic component of the full basin of attraction of  $\mathbf{a}$ ,  $\lambda$  is a  $J$ -stable parameter in  $\Lambda$ . We therefore choose not to call these parameters Misiurewicz parameters.

By definition, Misiurewicz parameters are solutions of

$$f_\lambda^m(s(\lambda)) = f_\lambda^n(s(\lambda)) \tag{5.1}$$

for some  $m > n \in \mathbb{N}$  and some singular value function  $s(\lambda)$ . If  $f_\lambda$  is conjugate to a Misiurewicz map  $f_{\lambda'}$  which satisfies Equation (5.1) for certain values of  $m, n \in \mathbb{N}$ , then  $\lambda$  satisfies the same equation for the same  $m, n$ .

Misiurewicz parameters exist in dynamically natural slices of functions. A meromorphic function that is not entire must have at least one pole that is not an omitted value. The pre-images of the poles have a finite forward orbit and play an important role in the dynamics.

In the remainder of this section let  $\mathcal{F} = \{f_\lambda, \lambda \in \Lambda\}$  be a dynamically natural slice of meromorphic maps and let  $v_\lambda$  denote the free asymptotic value.

**Definition 5.2 (Order of a prepole).** A point  $p \in \mathbb{C}$  is a *prepole of order*  $n > 0$  for  $f_\lambda$  if  $f_\lambda^k(p)$  is defined for  $k < n$  and  $f_\lambda^n(p) = \infty$ .

With this definition a pole is a prepole of order 1.

For every  $\lambda \in \Lambda$ , the poles form a discrete set so they can only accumulate at infinity. Unless there are at most two poles, and both are omitted values (and this is never the case for functions of finite type), Picard's theorem implies that the prepoles of order 2 form an infinite set. Their accumulation set is the set of poles and the point at infinity. Prepoles of order  $n \geq 3$  accumulate both at prepoles of order  $n - 1$  and at infinity. It is well known that prepoles are dense in the Julia set of  $f_\lambda$  [Ber93, BKY91].

We now look at the parameter plane and consider parameters for which some iterate of the free asymptotic value is a pole.

**Definition 5.3 (Virtual cycle).** Let  $f \in S$  be a meromorphic map. A *virtual cycle* for  $f$  of period  $p \geq 2$  is a set of points  $\{a_1, a_2, \dots, a_{p-1}, \infty\}$  where  $a_1$  is an asymptotic value,  $a_{p-1}$  is a pole and  $f(a_i) = a_{i+1}$  for  $1 \leq i \leq p - 2$ . In other words a virtual cycle is the forward orbit of an asymptotic value which is a prepole.

Note that it does not make sense to define virtual cycles of period one.

**Definition 5.4 (Virtual cycle parameter).** A parameter  $\lambda \in \Lambda$  is called a *virtual cycle parameter of order*  $p \geq 2$  if  $f_\lambda$  has a virtual cycle of period  $p$ .

For  $\lambda \in \Lambda$ , let  $\{p_i(\lambda)\}_{0 \leq i \leq M}$  denote the set of poles of  $f_\lambda$ , for some  $M \leq \infty$ . Recall that since we are working in a dynamically natural slice,  $p_i(\lambda)$  are distinct holomorphic functions of  $\lambda$  throughout  $\Lambda$ . By definition, virtual cycle parameters of order  $p \geq 2$  are  $\lambda$ -values which are solutions of

$$f_\lambda^{p-2}(v(\lambda)) = p_i(\lambda) \quad (5.2)$$

for some  $0 \leq i \leq M$ .

*Remark.* Virtual cycle parameters are parameter values for which an orbit relation exists. Hence if  $\lambda$  is a virtual cycle parameter and  $f_\lambda$  and  $f_{\lambda'}$  are topologically conjugate, then  $\lambda'$  must be a virtual cycle parameter of the same order.

Observe that in forming the orbit of the asymptotic value for a virtual cycle parameter, the iteration stops at infinity. This is in distinction to the orbit of the relevant singular value for a Misiurewicz parameter where the iteration does not stop, but the orbit contains only finitely many distinct values in  $\mathbb{C}$ . As a consequence, the sets of Misiurewicz parameters and virtual cycle parameters are disjoint.

**Proposition 5.5.** *Misiurewicz parameters and virtual cycle parameters belong to the bifurcation locus  $\mathcal{B}$ .*

*Proof.* Virtual cycle parameters are solutions of (5.2) while Misiurewicz parameters are solutions of (5.1). By the Identity Theorem, for fixed  $m, n$  and  $p$  both equations have a discrete set of solutions. Hence, since both conditions are invariant under topological conjugacy, the maps must be structurally unstable.  $\square$

**Proposition 5.6.** *Virtual cycle parameters are dense in  $\mathcal{B}(v_\lambda)$ , while Misiurewicz parameters are dense in  $\mathcal{B}$ . A consequence of both these statements is that  $\mathcal{B}(v_\lambda)$  has no isolated points in  $\Lambda$ .*

*Proof.* Using standard arguments, we first show that the virtual cycle parameters are dense in  $\mathcal{B}(v_\lambda)$ . Let  $\lambda_0 \in \mathcal{B}(v_\lambda)$  and let  $U$  be a neighborhood of  $\lambda_0$  in  $\Lambda$ . Let  $p_i(\lambda)$ ,  $i = 1, 2, 3$  be three prepoles of order 3 which are distinct for every  $\lambda \in U$ . These exist since the set of prepoles of order 3 or larger is infinite. Now observe that, for some  $\lambda \in U$ ,  $g_n(\lambda) = f_\lambda^n(v_\lambda)$  must take one of the values  $p_i(\lambda)$ , for some  $i = 1, 2$  or  $3$  and some  $n$ , because otherwise  $\{g_n\}$  would be normal in  $U$ , contradicting the assumption that  $\lambda_0 \in \mathcal{B}(v_\lambda)$ .

To show that Misiurewicz parameters are dense we use a repelling periodic point  $a(\lambda)$  of period larger than or equal to 3 for all  $\lambda \in U$ , and apply the above arguments to the functions  $\{a(\lambda), f_\lambda(a(\lambda)), f_\lambda^2(a(\lambda))\}$ .  $\square$

## 6 Shell components

Let  $\Lambda$  be a dynamically natural slice for the family  $\mathcal{F} = \{f_\lambda\}$  and let  $v_\lambda$  denote the free asymptotic value of  $f_\lambda$  which depends affinely on  $\lambda$ .

We are interested in studying those hyperbolic components in the slice  $\Lambda$  that reflect the behavior of the free asymptotic value. To define these components precisely, we first define the set

$$\mathcal{H} := \{\lambda \in \Lambda \mid v_\lambda \text{ is attracted to an attracting cycle with nonconstant multiplier}\}.$$

Observe that if  $\lambda \in \mathcal{H}$ , then  $f_\lambda$  is a hyperbolic map, since all remaining singular values must belong to the  $N - 1$  attracting basins of cycles with constant multiplier. As a consequence,

$$\mathcal{H} \subset \Lambda \setminus \mathcal{B} = \Lambda^{stab}.$$

**Definition 6.1 (Shell component).** A set  $\Omega \subset \Lambda$  is a *shell component* of the family  $f_\lambda$  if it is a connected component of  $\mathcal{H}$ .

Since  $\partial\Omega$  is in  $\mathcal{B}$ , it follows that  $\Omega$  is also a connected component of  $\Lambda \setminus \mathcal{B}$ .

Although shell components as defined above are always hyperbolic components in the classical sense, we have chosen to call them shell components for two reasons. On the one hand they are a very special type of hyperbolic component (for example, capture components, where the attracting cycle attracts more than one independent singular value, are hyperbolic but not shell). On the other hand, if the dynamically natural slice is defined in a relaxed sense, as explained in Remark ??, then maps belonging to a shell component would not be necessarily hyperbolic. Finally, we shall see that their boundaries have scalloped edges like shells.



The following property follows directly from the definition.

**Lemma 6.2 (The period is constant).** *If  $\Omega$  is a shell component, the period of the attracting cycle to which  $v_\lambda$  is attracted is constant throughout  $\Omega$ . It is called the period of  $\Omega$ .*

Since the dynamics of all but one of the attracting cycles of  $f_\lambda$  are fixed as  $\lambda$  varies in a shell component, *the attracting cycle* for this shell component is the one whose *multiplier* varies with  $\lambda$ . Below, when we talk about an attracting cycle or its multiplier, in relation to a shell component, it is these that we mean.

Every hyperbolic component of the parameter plane of the well known exponential family  $E_\lambda = z \mapsto e^z + \lambda$  is an example of a shell component. In this case, the multiplier of the attracting cycle of  $E_\lambda$  is never zero. This also happens in more general settings.

**Lemma 6.3 (Nonzero multipliers).** *If  $\Omega$  is a shell component, the multiplier of the attracting cycle is nonzero for all  $\lambda \in \Omega$ .*

*Proof.* If the multiplier of the attracting cycle is zero, the cycle contains a critical point. Hence  $v_\lambda$  is not the only singular value in the basin of attraction of this cycle, contradicting the definition of a shell component.  $\square$

The following theorem explains how shell components are parameterized by the multiplier of the attracting cycle. Let  $\Omega$  be a shell component of period  $k > 0$ . We define the multiplier map

$$\rho = \rho_\Omega : \Omega \rightarrow \mathbb{D}^*$$

where  $\rho(\lambda)$  is the multiplier of the attracting cycle which attracts  $v_\lambda$ .

**Theorem 6.4 (Multiplier map is a covering).** *The multiplier map  $\rho : \Omega \rightarrow \mathbb{D}^*$  is a holomorphic covering.*

*Proof.* The proof uses the method of *surgeries* on the basin of attraction of the attracting cycle that attracts  $v_\lambda$ . For details see [BF14]. Observe that because  $f_\lambda$  is a holomorphic family both  $\rho$  and its derivative are holomorphic functions of  $\lambda$ .

Choose a parameter  $\lambda_0 \in \Omega$  such that  $\omega_0 := \rho(\lambda_0) \in \mathbb{D}^*$ . Let  $k > 0$  be the period of  $\Omega$  and  $\mathbf{a}^0 = \{a_0^0, \dots, a_{k-1}^0\}$  the attracting periodic orbit to which the free asymptotic value  $v_{\lambda_0}$  is attracted. Let  $\mathcal{A}^0$  denote its basin of attraction and let  $A_i^0$  be the connected component of  $\mathcal{A}^0$  containing  $a_i^0$  for  $i = 0, \dots, k-1$ . Assume without loss of generality that  $v_{\lambda_0} \in A_1^0$ .

Let  $W \subset \mathbb{D}^*$  be a (simply connected) neighborhood of  $\omega_0$ . For any  $\omega \in W$  we define a Beltrami form  $\mu_\omega$  in  $A_1^0$  invariant under  $f_{\lambda_0}^k$  as follows. Since  $\mathbf{a}^0$  is attracting, there is a holomorphic homeomorphism  $\psi$  from a neighborhood  $U$  of  $a_1^0$  to the unit disk  $\mathbb{D}$  such that  $\psi(a_1^0) = 0$  and  $\psi(f_{\lambda_0}^k(z)) = \omega_0 \psi(z)$  for every  $z \in U$ . In  $\mathbb{D}$  choose an annular fundamental domain  $R_0 = \{|\omega_0| < |z| < 1\}$  for the map  $\psi(z) \mapsto \omega_0 \psi(z)$ . Now we define the surgery using standard arguments: Construct a quasiconformal map  $\phi_\omega : \mathbb{D} \rightarrow \mathbb{D}$  such that it fixes the origin and the unit circle and it maps  $R_0$  to  $R_\omega = \{|\omega| < |w| < 1\}$  conjugating multiplication by  $\omega_0$  to multiplication by  $\omega$ . Additionally, we require that  $\tilde{\phi}_{\omega_0} = Id$  and that  $\tilde{\phi}_\omega$  depends holomorphically on  $\omega$ . To do this, first define the map on  $R_0$  and then extend it by the

dynamics to all of  $\mathbb{D}$ . The Beltrami coefficient  $\tilde{\mu}_\omega$  of  $\tilde{\phi}_\omega$  lifts to a Beltrami coefficient  $\mu_\omega$  on  $U$  invariant under the map  $\psi$  that satisfies  $\mu_\omega(f_{\lambda_0}^k(z)) \frac{(f_{\lambda_0}^k)'(z)}{(f_{\lambda_0}^k)'(z)} = \mu_\omega(z)$ . Extend  $\mu_\omega$  to the full backward orbit of  $U$  so that for any inverse branch  $g_n$  of  $f_{\lambda_0}^{-n}$  defined on a component of  $\mathcal{A}^0$ ,  $\mu_\omega(g_n(z)) \frac{g_n'(z)}{g_n'(z)} = \mu_\omega(z)$ . Extend it to be identically zero on the complement of the full basin of  $\mathbf{a}^0$  in  $\mathbb{C}$ . A Beltrami coefficient with this property is said to be *compatible with  $f_{\lambda_0}$* . Notice that the Beltrami coefficient  $\mu_\omega$  depends holomorphically on  $\omega$ .

By the Measurable Riemann Mapping Theorem [tt60], there is a quasiconformal homeomorphism  $\phi_\omega : \mathbb{C} \rightarrow \mathbb{C}$  with Beltrami coefficient  $\mu_\omega$ . It is unique up to post-composition by an affine map and depends holomorphically on  $\omega$ . Moreover,  $\phi_{\omega_0}$  is conformal.

At this point we need to assume that the functions in the family  $f_\lambda$  are normalized in the same way by, for example, having two marked points at 0 and 1. Then we can chose a uniform normalization for  $\phi_\omega$  fixing these two points and define the new meromorphic map

$$F_\omega := \phi_\omega \circ f_{\lambda_0} \circ \phi_\omega^{-1}.$$

This map respects the dynamics: it has an attracting periodic cycle  $\mathbf{a}(\omega) = \phi_\omega(\mathbf{a}^0)$  of multiplier  $\omega$ . Moreover,  $F_\omega$  is quasiconformally conjugate to  $f_{\lambda_0}$  in the respective basins of attraction and conformally conjugate to  $f_{\lambda_0}$  everywhere else. Since  $\phi_{\omega_0}$  is conformal and fixes two points it follows that it is the identity map. Hence  $F_{\omega_0} = f_{\lambda_0}$ .

Since  $f_\lambda$  is a dynamically natural slice, property (d) ensures that  $F_\omega = f_{\lambda(\omega)}$  for some  $\lambda(\omega) \in \Lambda$ . Observe that the asymptotic value of  $f_{\lambda(\omega)}$  is  $v_\omega = \phi_\omega(v_{\lambda_0})$  and it depends holomorphically on  $\omega$ . Since  $v_\lambda = v(\lambda)$  is affine,  $\lambda = \lambda(v)$  is also affine and hence  $\lambda(\omega) = \lambda(v_\omega)$  is holomorphic. Since the multiplier changes with  $\omega$ , the map is nonconstant and thus open. Finally,  $\lambda(\omega_0) = \lambda_0$  and therefore  $\lambda(W) \subset \Omega$ .

This construction defines a holomorphic local inverse of the multiplier map  $\rho$  in a neighborhood of any point  $\omega_0 = \rho(\lambda_0) \in \mathbb{D}^*$ . Since  $\rho(\lambda) \neq 0$  for all  $\lambda \in \Omega$ , by Lemma 6.3, it follows that  $\rho$  is a covering map from  $\Omega$  to  $\mathbb{D}^*$ .  $\square$

**Corollary 6.5 (Connectivity of shell components).** *Let  $\Omega$  be a shell component of  $f_\lambda$ . Then one of the two following possibilities occurs:*

- (a)  $\Omega$  is simply connected and  $\rho : \Omega \rightarrow \mathbb{D}^*$  is a universal covering (hence of infinite degree), or
- (b)  $\Omega$  is isomorphic to  $\mathbb{D}^*$  and  $\lim_{|\omega| \rightarrow 0} \rho^{-1}(\omega) = \lambda^*$  where  $\lambda^* \notin \Lambda$  is a parameter singularity.

*Proof.* Since  $\rho : \Omega \rightarrow \mathbb{D}^*$  is a covering, it follows from Theorem 2.5 that either  $\rho$  is a universal covering and hence has infinite degree and  $\Omega$  is simply connected, or  $\rho$  has finite degree and  $\Omega$  is isomorphic to  $\mathbb{D}^*$ . In this case, the point  $\lambda^* := \lim_{|\omega| \rightarrow 0} \rho^{-1}(\omega)$  cannot belong to  $\mathcal{B}$  because, by Proposition 5.6,  $\mathcal{B}$  has no isolated points. It follows from Lemma 6.3 that  $\rho$  cannot extend continuously to  $\lambda^*$  because  $f_\lambda$  has no superattracting cycles. Therefore  $\lambda^*$  is a parameter singularity.  $\square$

**Corollary 6.6.** *Let  $\Omega$  be a shell component. Then, any two maps  $f_\lambda$  and  $f_{\lambda'}$  with  $\lambda, \lambda' \in \Omega$  are (globally) quasiconformally conjugate.*

*Proof.* This follows from the proof of Theorem 6.4. Indeed, let  $\gamma : [0, 1] \rightarrow \Omega$  be a continuous path with endpoints  $\lambda$  and  $\lambda'$ . Then  $\sigma(t) := \rho(\gamma(t))$  is a continuous path in  $\mathbb{D}^*$  joining  $\omega = \rho(\lambda)$  and  $\omega' = \rho(\lambda')$ . The image of  $\sigma$  is a compact set so it can be covered by a finite number of open disks  $\{D_i\}_{1 \leq i \leq N}$  in  $\mathbb{D}^*$  each centered at a point  $\sigma(t_i)$ . Using the construction in Theorem 6.4, we can construct local inverses of  $\rho$  mapping  $D_i$  to an open neighborhood  $D'_i$  of  $\gamma_{t_i}$ . We leave it to the reader to check that the disks can be chosen so that  $\gamma[0, 1] \subset \cup_{i=1}^N D'_i$ . Since all pairs of parameters in each  $D'_i$  correspond to quasiconformally conjugate maps, it follows that  $f_\lambda$  is quasiconformally conjugate to  $f_{\lambda'}$ .  $\square$

## 6.1 Virtual centers

In standard rational dynamics hyperbolic components have a distinguished point called *the center* of the component at which the attracting cycle is superattracting. This is not the case for the shell components we are studying, where the multiplier of a finite cycle cannot vanish. Nevertheless we shall see that there are points in the boundary of the shell component that play the role of the center.

**Definition 6.7 (Virtual center).** Let  $\Omega$  be a shell component and  $\rho : \Omega \rightarrow \mathbb{D}^*$  its multiplier map. A point  $\lambda \in \partial\Omega \cap \Lambda$  is a *virtual center* of  $\Omega$  if whenever  $\lambda_n \rightarrow \lambda$ , with  $\lambda_n \in \Omega$ , the multipliers  $\rho(\lambda_n)$  tend to 0.

In this section we will restrict our attention to dynamically natural slices  $\Lambda$  of maps in  $\mathcal{M}_\infty$ . Other cases like families of entire maps, have been studied previously in a more direct way. More precisely, let  $\{f_\lambda\}_{\lambda \in \Lambda}$  be a dynamically natural slice of a holomorphic family of finite type meromorphic maps for which infinity is not an asymptotic value. Let  $v(\lambda)$  denote the free asymptotic value.

For maps in this class, we show that every shell component has a unique virtual center which may be finite or infinite. In addition, we show that if the virtual center of a shell component is finite then it is a virtual cycle parameter. To this end, we need some preliminary results.

The following two technical propositions describe the possibilities for what happens in the dynamical plane when we take a sequence of parameters in a shell component approaching a finite boundary point. The first one deals with the case where the cycle stays bounded, and applies to both entire and meromorphic functions.

**Proposition 6.8 (Approaching the boundary I).** *Let  $\Omega$  be a shell component of period  $k \geq 1$  with  $\lambda_n \in \Omega$ , and suppose  $\lambda_n \rightarrow \lambda^* \in \partial\Omega \cap \Lambda$ . Let  $\mathbf{a}^n = \{a_0^n, \dots, a_{k-1}^n\}$  be the attracting cycle of period  $k$  for  $f_{\lambda_n}$ , labeled so that  $a_1^n$  belongs to the component of the basin containing the asymptotic value  $v_n := v(\lambda_n)$ , and so that  $a_{i+1}^n = f_{\lambda_n}(a_i^n)$  (with indices taken mod  $k$ ). Suppose the cycle  $\mathbf{a}^n$  stays bounded as  $n \rightarrow \infty$ . Then,  $f_{\lambda^*}$  has an indifferent cycle  $\mathbf{a}$  whose period divides  $k$  and  $\mathbf{a}^n \rightarrow \mathbf{a}$  as  $n \rightarrow \infty$ .*

*Proof.* Since the cycle stays bounded and  $\lambda^* \in \Lambda$ , it follows that the sequence  $a_i^n$  has a limit  $a_i$  for all  $0 \leq i \leq k-1$ , and that this limit point is a fixed point of  $f_{\lambda^*}^k$ . This cycle must be non-repelling. Since  $\lambda^* \in \partial\Omega$ , it cannot be attracting. Hence it is an indifferent cycle.  $\square$

**Proposition 6.9 (Approaching the boundary II).** *Let  $\Omega$  be a shell component of period  $k \geq 2$  with  $\lambda_n \in \Omega$ , and suppose  $\lambda_n \rightarrow \lambda^* \in \partial\Omega \cap \Lambda$ . Let  $\mathbf{a}^n = \{a_0^n, \dots, a_{k-1}^n\}$  be the attracting cycle of period  $k$  for  $f_{\lambda_n}$ , labeled so that  $a_1^n$  belongs to the component of the basin containing the asymptotic value  $v_n := v(\lambda_n)$ , and so that  $a_{i+1}^n = f_{\lambda_n}(a_i^n)$  (with indices taken mod  $k$ ). Suppose  $|a_j^n| \rightarrow \infty$  as  $n \rightarrow \infty$ , for some  $0 \leq j \leq k-1$ . Then,  $j = 0$  and as  $n \rightarrow \infty$ ,*

- (a)  $a_1^n \rightarrow v(\lambda^*)$ , the free asymptotic value of  $f_{\lambda^*}$ ;
- (b)  $a_{k-1}^n$  tends to a pole of  $f_{\lambda^*}$ ;
- (c)  $a_{k-i}^n$  tends to a prepole of order  $i$  for all  $1 \leq i \leq k-1$ ;
- (d) The multiplier of the cycle tends to 0.

Consequently,  $\lambda^*$  is a virtual cycle parameter. Moreover, it is also a virtual center.

*Proof.* Observe that  $f_{\lambda^*}$  is a well defined map in our family because  $\lambda^* \in \Lambda$ . Let  $D$  be a closed disk containing all the singular values of  $f_{\lambda_n}$  for all  $n$  sufficiently large. Since infinity is not an asymptotic value, by Lemma 3.3 the preimage of  $\mathbb{C} \setminus D \simeq \mathbb{D}^*$  under  $f_{\lambda_n}$  consists of countably many bounded topological punctured disks  $\{B_n^i\}_i$ . The punctures correspond to distinct poles  $p_n^i$  and  $f_{\lambda_n} : B_n^i \cup \{p_n^i\} \rightarrow \widehat{\mathbb{C}} \setminus D$  is a finite degree branched cover. Infinitely many of these punctured disks must be entirely contained in  $\mathbb{C} \setminus D$ , and provided that  $D$  is large enough, they are disjoint. If  $p_n^i$  is a simple pole, there is a single valued branch of the inverse of  $f_{\lambda_n}$  that maps neighborhood of infinity in  $\mathbb{C}$  onto  $B_n^i$ . If the pole is multiple then there are finitely many such inverse branches.

Note that since we are in a dynamically natural slice, the poles  $p_n^i = p^i(\lambda_n)$  of  $f_{\lambda_n}$  are holomorphic functions of  $\lambda$ . It follows that because  $\lambda_n$  converges to a finite point  $\lambda^* \in \Lambda$ , the functions  $p^i(\lambda_n)$  converge to the poles  $p^i(\lambda^*)$  of  $f_{\lambda^*}$ . Therefore if we take  $n$  sufficiently large (and assume we have indexed the disks consistently), the disks  $\{B_n^i\}_i$  and the poles  $p_n^i$  are arbitrarily close (in the spherical metric) to  $B_*^i$  and  $p^i(\lambda^*)$ , the analogous sets and points for  $f_{\lambda^*}$ . Let  $U := \mathbb{C} \setminus \cup_i \overline{B_*^i}$ .

By hypothesis, for some  $0 \leq j \leq k-1$ ,  $|a_j^n| \rightarrow \infty$  as  $n \rightarrow \infty$ . We first observe that  $a_{j+1}^n = f_{\lambda_n}(a_j^n)$  stays bounded for all  $n \geq 0$ . Indeed if it tends to infinity, for every  $n$  large enough,  $a_j^n$  belongs to a disk  $B_n^i$  for some  $i$ . But since  $a_j^n$  tends to infinity with  $n$  it follows that  $i$  must change with  $n$ . This is equivalent to saying that the inverse branch of  $f_{\lambda_n}^{-1}$  mapping  $a_{j+1}^n$  to  $a_j^n$  changes as  $n$  increases. But inside the shell component  $\Omega$  all parameters can be joined by a path where the inverse branches are well defined and quasiconformally conjugate and thus cannot change.

Hence  $a_{j+1}^n$  converges to a finite point  $a_{j+1}$ . We now prove that  $a_{j+1}$  is an asymptotic value of  $f_{\lambda^*}$ . Suppose it is not. Since  $f_{\lambda}$  is of finite type for all  $\lambda$ , there exists a small neighborhood  $V$  of  $a_{j+1}$  such that  $f_{\lambda^*}^{-1}(a_{j+1})$  consists of infinitely many bounded disjoint topological disks  $\{D_i\}_i$ . Let  $N$  be such that  $a_{j+1}^n \in V$  for  $n > N$ . Then the points  $f_{\lambda^*}^{-1}(a_{j+1}^n)$

are contained in the disks  $D_i$ . Since the inverse branch taking  $a_{j+1}^n$  to  $a_j^n$  cannot change within  $\Omega$ , it follows that the points in the attracting cycle must actually lie in a single one of these disks, which contradicts the hypotheses that  $a_j^n \rightarrow \infty$  when  $n \rightarrow \infty$ .

We now know that  $a_{j+1}$  is an asymptotic value of  $f_{\lambda^*}$ . Since the singular values of  $f_{\lambda}$  are holomorphic functions of  $\lambda$ , there is holomorphic map determining an asymptotic value  $u(\lambda)$  of  $f_{\lambda}$  such that  $u(\lambda^*) = a_{j+1}$ . Let  $T^* = T(\lambda^*)$  be the asymptotic tract of  $u(\lambda^*)$  which contains  $a_j^n$  for  $n$  large enough (which exists because  $a_{j+1}^n \rightarrow u(\lambda^*)$ ). Let  $T_n$  be the corresponding asymptotic tracts for  $u_{\lambda_n}$ . Then  $T^*$  and  $T_n$  contain a non-empty domain  $T'$  such that  $a_j^n \in T'$ . If  $V' = f_{\lambda^*}(T')$ , it follows that  $V'$  contains  $a_{j+1}^n$  and  $u(\lambda^*)$ ; moreover, since  $T'$  is contained in the asymptotic tracts  $T_n$  it follows that  $u(\lambda_n)$  also lie in  $V'$ .

We claim that  $u(\lambda_n)$  lies in the component of the immediate basin containing  $a_{j+1}^n$  when  $n$  is large enough. By the definition of a dynamically natural slice,  $v(\lambda_n)$  is the only asymptotic value in the basin of  $a_{j+1}^n$  and there are  $N - 1$  other attracting cycles with fixed multiplier. Each of these attracts a singular value. Thus if  $u(\lambda_n)$  were outside the basin, it would be attracted to one of the other  $N - 1$  attracting cycles that persist throughout the slice and thus in a different basin. Taking a pre-image of a neighborhood of  $u(\lambda_n)$  inside this basin, we see that any asymptotic tract corresponding to  $u(\lambda_n)$  is mapped within the basin itself and so the basin could not contain  $a_j^n$  for any  $n$ . Therefore since  $v(\lambda_n)$  is the only asymptotic value in the basin of  $a_{j+1}^n$ ,  $u(\lambda_n) = v(\lambda_n)$  and the claim follows. Hence  $j + 1 = 1$  and  $j = 0$  and the limit of  $a_{j+1}^n$  as  $n \rightarrow \infty$  is  $v(\lambda^*)$ . This proves (a).

To show (b), observe that by continuity  $a_{k-1}^n$  is a sequence which must tend either to infinity or to a pole of  $f_{\lambda^*}$ , given that its image  $a_0^n$  is converging to infinity. As above, if it converges to a pole, this is a finite point and we are done. By the same arguments as above if it were to converge to infinity then  $a_0^n$  would converge to the asymptotic value and infinity is not an asymptotic value. Hence  $a_{k-1} := \lim_{n \rightarrow \infty} a_{k-1}^n$  is a pole of  $f_{\lambda^*}$  and (b) is proved.

If  $k = 2$  there is nothing to show for (c). If  $k > 2$ , using the same arguments as in (a) we can see that no other point of the cycle can be tending to infinity. Indeed, if  $a_i^n$  tends to infinity we have shown that  $v(\lambda_n)$  is in the component containing  $a_{i+1}^n$  and hence  $i = 0$ . Thus all  $a_i^n$  must converge to finite prepoles of the appropriate order. This shows that the cycle is virtual and hence  $\lambda^*$  is a virtual cycle parameter.

Finally we prove (d). The multiplier of the cycle for a given  $n$  is  $\rho_n = \prod_{i=0}^{k-1} |f'_{\lambda_n}(a_i^n)|$ . When  $n \rightarrow \infty$  all factors tend to a finite number except for  $i = 0$  or  $i = k - 1$ . In the first case we have that  $|f'_{\lambda_n}(a_0^n)| \rightarrow 0$  exponentially (since  $a_0^n$  is inside the asymptotic tract of  $v_{\lambda_n}$  for  $n$  large enough). In the second one,  $|f'_{\lambda_n}(a_{k-1}^n)| \rightarrow \infty$  polynomially (like  $1/|z_n - p|^m$  for  $m$  fixed). It follows that the product tends to 0 as  $n \rightarrow \infty$ .

Since  $\mathcal{F}$  is a holomorphic family, the functions depend continuously on  $\lambda$ . Therefore if we approach  $\lambda^*$  with a different sequence  $\lambda'_n \in \Omega$ , the attracting cycle of  $f_{\lambda'_n}$  must be arbitrarily close to the virtual cycle of  $f_{\lambda^*}$  as  $n$  tends to infinity. It follows that the multiplier is also tending to 0 and we have proved that  $\lambda^*$  is a virtual center.  $\square$

We are now ready to prove the main theorems in this section.

**Theorem 6.10 (Virtual centers and virtual cycles).** *Let  $\Lambda$  be a dynamically natural slice for a family of meromorphic maps  $\{f_{\lambda}, \lambda \in \Lambda\}$  in  $\mathcal{M}_{\infty}$ . Let  $\Omega \subset \Lambda$  be a shell component*

of period  $k \geq 2$  and  $\lambda^*$  a parameter in  $\partial\Omega \cap \Lambda$ . Then  $\lambda^*$  is a virtual center if and only if  $\lambda^*$  is a virtual cycle parameter.

*Proof.* Assume  $\lambda^*$  is a virtual center. Let  $\lambda_n \rightarrow \lambda^*$ , with  $\lambda_n \in \Omega$  and let  $\mathbf{a}_n = \{a_0^n, \dots, a_{k-1}^n\}$  be the cycle to which  $v_n = v(\lambda_n)$  is attracted for  $f_{\lambda_n}$ . We claim that one of the points in the cycle tends to infinity as  $n \rightarrow \infty$ . Indeed, otherwise, the cycle would converge to a finite cycle  $\mathbf{a}^*$  of the same period and multiplier 0. Hence  $\mathbf{a}^*$  is superattracting and some point in the limit cycle is a critical point. But this implies that for every  $\lambda$  in a neighborhood of  $\lambda^*$  the analytic continuation of this cycle is attracting and its basin contains a critical point. This contradicts the assumption that  $\lambda^*$  is in  $\partial\Omega$  and that some part of this neighborhood belongs to the shell component. It now follows from Proposition 6.9 that  $\lambda^*$  is a virtual cycle parameter.

Conversely, suppose that  $\lambda^*$  is a virtual cycle parameter, and let  $\lambda_n \rightarrow \lambda^*$  as above. Then, by the definition of a virtual cycle parameter, at least one of the points in the cycle is tending to infinity when  $n \rightarrow \infty$ . By Proposition 6.9, the multiplier is then tending to zero. Since the sequence was arbitrary,  $\lambda^*$  is a virtual center. □

The next proposition deals with the special case of shell components of period one.

**Proposition 6.11.** *Let  $\Lambda$  be a dynamically natural slice for a family of meromorphic maps  $\{f_\lambda, \lambda \in \Lambda\}$ . Let  $\Omega$  be a simply connected shell component of period one. Then, for any sequence  $\lambda_n \rightarrow \lambda^* \in \partial\Omega \cap \Lambda$ , the attracting fixed point  $a_n$  converges to an indifferent finite fixed point  $a^*$  of  $f_{\lambda^*}$ . It follows that  $\Omega$  has no finite virtual centers.*

*Remark 6.12.* Note that if  $\Omega$  is isomorphic to a punctured disk, the puncture is a parameter singularity and does not belong to  $\Lambda$  and the proposition does not apply.

*Proof.* Suppose that the attracting fixed point  $a_n$  for  $f_{\lambda_n}$  tends to infinity as  $n \rightarrow \infty$ . If  $f_\lambda$  is meromorphic and infinity is not an asymptotic value of any of the functions in our family we argue as in the proof of Proposition 6.9. Let  $D$  be a large disk eventually containing all the singular values of  $f_{\lambda_n}$  and let  $B_n^i$  be the components of  $f_{\lambda_n}^{-1}(\mathbb{C} \setminus D)$ . Since the  $a_n$  are fixed points and form an unbounded sequence, they must belong to  $B_n^i$  for  $n$  large enough. Because the maps  $f_{\lambda_n}$  are polynomially expanding on  $B_n^i$ , the derivative in these disks is larger than one, so there cannot be any attracting fixed points inside them. Therefore the  $a_n$  must converge to a finite fixed point which, by Proposition 6.8, is an indifferent fixed point. This shows the first part of the Proposition.

If the functions in our family have infinity as an asymptotic value, for example if the functions are entire, we choose the disk  $D$  so that it contains all the finite singular values of  $f_{\lambda_n}$  for large  $n$ . Then  $f_{\lambda_n}^{-1}(\mathbb{C} \setminus D)$  has at least one simply connected unbounded component that is an asymptotic tract for the asymptotic value infinity. In this tract the function is expanding exponentially so again it cannot contain an attracting fixed point.

Now suppose  $\lambda^* \in \partial\Omega \cap \Lambda$  is a virtual center, and let  $\lambda_n \rightarrow \lambda^*$  with  $\lambda_n \in \Omega$ . Since the attracting fixed point  $a_n$  of  $f_{\lambda_n}$  stays bounded, it must converge to a finite fixed point of multiplier 0. But this is impossible by Proposition 6.8. □

We are now ready to study the properties of the boundary of the simply connected shell components in more detail. Let  $\Omega$  be a simply connected shell component and let  $\mathbb{H}_l$  denote the left half plane. Recall from Corollary 6.5 that the multiplier map  $\rho : \Omega \rightarrow \mathbb{D}^*$  is a universal covering. Hence, there exists a conformal homeomorphism  $\varphi : \mathbb{H}_l \rightarrow \Omega$ , unique up to precomposition by a Möbius transformation, such that

$$(\rho \circ \varphi)(w) = e^w.$$

**Theorem 6.13.** *Let  $\Lambda$  be a dynamically natural slice for a family of meromorphic maps  $\{f_\lambda, \lambda \in \Lambda\}$  in  $\mathcal{M}_\infty$ . Let  $\Omega$  be a simply connected shell component in  $\Lambda$  and let  $\rho, \varphi$  be as above. Then,  $\varphi$  extends continuously to a map  $\varphi : \overline{\mathbb{H}_l} \cup \infty \rightarrow \overline{\Omega}$ , where the closure is taken in  $\widehat{\mathbb{C}}$ .*

*Proof.* Let  $\gamma(t)$ ,  $t \in (0, 1)$  be an arbitrary arc in  $\mathbb{H}_l$  such that  $\lim_{t \rightarrow 1} \gamma(t) = i\theta^*$  for some  $\theta^* \in \mathbb{R}$ . We want to show that  $\sigma(t) = \varphi(\gamma(t))$  lands at a point in  $\overline{\Omega} \setminus \Omega$  when  $t \rightarrow 1$ .

To this end, let  $\lambda^*$  be a finite accumulation point of  $\sigma(t)$  when  $t \rightarrow 1$ . This means that there exists a sequence  $t_n \rightarrow 1$  as  $n \rightarrow \infty$ , such that  $\lambda_n = \sigma(t_n) \in \Omega$  satisfies  $\lambda_n \rightarrow \lambda^*$ . Let  $\mathbf{a}^n$  denote the attracting cycle corresponding to  $\lambda_n$ . If one point in this cycle goes to infinity as  $n \rightarrow \infty$ , it follows from Proposition 6.9 that the multiplier tends to zero which is impossible. Therefore the cycle stays bounded and, by Proposition 6.8, it tends to an indifferent cycle of multiplier  $e^{2\pi i \theta^*}$ , and period dividing the period of  $\Omega$ . By the Identity Theorem, the parameters for which the function has a periodic cycle with this multiplier and period either form a discrete set or constitute the whole slice  $\Lambda$ . They cannot fill out the whole slice because by the definition of a dynamically natural slice, the singular value that can accumulate on such a neutral cycle must be the free asymptotic value  $v_\lambda$ , and if  $\lambda \in \Omega$  the cycle is attracting. Since the accumulation set of  $\sigma(t)$  must be connected it cannot be a discrete set. It then follows that  $\sigma(t)$  actually lands at  $\lambda^* \in \partial\Omega$ .

We conclude that  $\sigma$  lands either at infinity or at a finite parameter value  $\lambda^*$  for which  $f_{\lambda^*}$  has an indifferent cycle. It follows from the Lindelöf theorem [CG93, Theorem 2.2] that the image of every other possible arc  $\gamma$  approaching  $i\theta^*$  must have the same landing point as  $\gamma$ .

It remains to consider curves in  $\mathbb{H}_l$  that tend to infinity. First suppose  $\gamma(t) = t$ ,  $t \in (-\infty, 0)$ , the negative real axis. We claim that  $\sigma(t) = \varphi(\gamma(t))$  lands at a point in  $\partial\Omega \cap \mathbb{C}$  or at infinity. Indeed, let  $\lambda^* \in \partial\Omega$  be a finite accumulation point of  $\sigma$ . As above, there exists a sequence  $t_n \rightarrow -\infty$  as  $n \rightarrow \infty$ , such that  $\lambda_n = \sigma(t_n) \in \Omega$  and  $\lambda_n \rightarrow \lambda^*$ . Let  $\mathbf{a}^n$  denote the attracting cycle corresponding to  $\lambda_n$ . If all the points in the cycle stay bounded as  $n$  grows then, by Proposition 6.8, the multiplier of the cycle tends to one in modulus, which contradicts the fact that  $\exp(\gamma(t))$  is tending to 0. Hence at least one point of the cycle must tend to infinity and, by Proposition 6.9,  $\lambda^*$  is a virtual cycle parameter of order equal to the period of  $\Omega$ , and a virtual center. But virtual cycle parameters of a given period form a discrete set. Thus all the accumulation points of  $\sigma$  must equal  $\lambda^*$  which implies that  $\sigma$  actually lands at  $\lambda^*$ .

The considerations above imply that the radial limit of  $\varphi$  at infinity exists, and in fact, by the Lehto-Virtanen Theorem [Pom92, Sect 4.1], the angular limit exists (i.e., the limit for any curve tending to infinity in a Stolz angle), and equals the virtual center  $\lambda^*$  or infinity.

If we take any other curve in  $\mathbb{H}_l$  tending to infinity, Lindeloff's theorem implies that any accumulation point must equal the radial limit.

We conclude that unrestricted limits exist at all finite points of  $\partial\mathbb{H}_l$ , and the radial limit exists at infinity. Hence  $\phi$  extends continuously to the boundary in  $\widehat{\mathbb{C}}$ .  $\square$

An immediate corollary of the proof is the following.

**Corollary 6.14.** *Let  $\Lambda$  be a dynamically natural slice for a family of meromorphic maps  $\{f_\lambda, \lambda \in \Lambda\}$ . Let  $\Omega$  be a simply connected shell component in  $\Lambda$ . Then,  $\partial\Omega \subset \widehat{\mathbb{C}}$  is locally connected and has a unique virtual center, which may be at infinity.*

Combining the existence of a unique virtual center with Proposition 6.11, we obtain the following immediate corollary of Theorem 6.13.

**Corollary 6.15.** *Under the hypotheses of the corollary above, every simply connected shell component of period 1 has its virtual center at infinity and it is therefore unbounded.*

*Remark 6.16.* If a shell component is not simply connected, the role of the virtual center may be taken by a parameter singularity. An example can be found in the tangent family  $z \mapsto \lambda \tan(z)$ , where  $\mathbb{D}^*$  is a shell component of period one with a parameter singularity in its center. In this case, the multiplier map is a conformal homeomorphism.

Theorem 6.13 also allows us to define an internal structure in the simply connected shell component  $\Omega$ .

**Corollary 6.17.** *Under the hypotheses of Corollary 6.14, define the internal ray in  $\Omega$  of angle  $\theta$  by*

$$R_\Omega(\theta) := \{\varphi(t + 2\pi i\theta), t \in (-\infty, 0)\}.$$

*Then, all internal rays have one end at the virtual center and the other end at a point in  $\partial\Omega$  (which a priori could be infinity). If the virtual center  $\lambda^*$  is finite no internal ray has both ends at  $\lambda^*$ .*

Notice that the internal rays foliate the shell component. The rays landing at parameters corresponding to parabolic cycles of multiplier one, that is the preimages of  $\{0, 1\}$  under the multiplier map extended to  $\partial\Omega$ , divide  $\Omega$  into fundamental domains which are mapped one to one to the punctured disk by the multiplier map  $\rho$ .

We have shown that simply connected shell components of period one are always unbounded. In the next section we look at some examples of dynamically natural slices of parameter space for specific families. The first is the exponential family  $\{E_\lambda = e^z + \lambda, \lambda \in \mathbb{C}\}$  where all components are unbounded shell components. In the other examples, the families of meromorphic maps  $\{f_\lambda, \lambda \in \Lambda\}$  are in  $\mathcal{M}_\infty$  and in all of these we see that all of the simply connected shell components of period greater than one are bounded. This leads us to the following conjecture.

**Conjecture 6.18.** *Let  $\Lambda$  be a dynamically natural slice for a family of meromorphic maps  $\{f_\lambda, \lambda \in \Lambda\}$  in  $\mathcal{M}_\infty$ . Let  $\Omega \subset \Lambda$  be a simply connected shell component of period  $k \geq 2$ . Then  $\Omega$  is bounded.*



## 7 Examples revisited

In Section 4.1 we looked at a number of different examples of dynamically natural slices of transcendental functions of finite type. In this section we revisit these examples in order to see the structure of the shell components inside the slices. The numbering of the examples here agrees with the numbering in Section 4.1.

**Example 1.** In this first set of examples the functions are all in  $\mathcal{F}_2$ ; that is they have two asymptotic values and no critical values.

- (a) Figure 1 shows the parameter plane for the family of entire maps  $E_\lambda(z) = \exp z + \lambda$ ,  $\Lambda = \{\lambda \in \mathbb{C}\}$ . The functions in this well studied family have one finite asymptotic value at  $v_\lambda = \lambda$  and no critical points and, although they were not called shell components, it was shown that all components of  $\mathcal{H} \subset \Lambda$  are unbounded and simply connected and that the multiplier goes to zero as the parameter goes to infinity inside the component; that is the components have a virtual center at infinity. The functions in this family have no poles so there are no virtual cycle parameters. Observe that the shell components are all unbounded, even those of period greater than 1. Components are colored according to their respective period, yellow for 1, cyan-blue for 2, red for 3, brownish green for 4, etc.

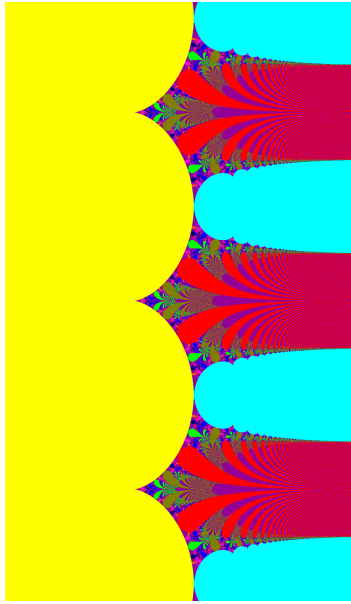


Figure 1: The parameter plane of the exponential family  $E_\lambda(z) = \exp(z) + \lambda$ .

- (b) Figure 2 shows the parameter plane for the dynamically natural slice of  $\mathcal{F}_2$  defined by keeping the origin as a fixed point with multiplier  $\rho_0 = 2/3$ . The functions have the form

$$f_\lambda(z) = \frac{e^z - e^{-z}}{(1/\lambda)e^z + (1/\mu)e^{-z}}$$

with  $\mu = -\lambda/(1 - 3\lambda)$ . The asymptotic values are at  $\lambda$  and  $-\mu$ . There are parameter singularities at 0 and  $1/3$  so that  $\Lambda = \mathbb{C} \setminus \{0, 1/3\}$ .

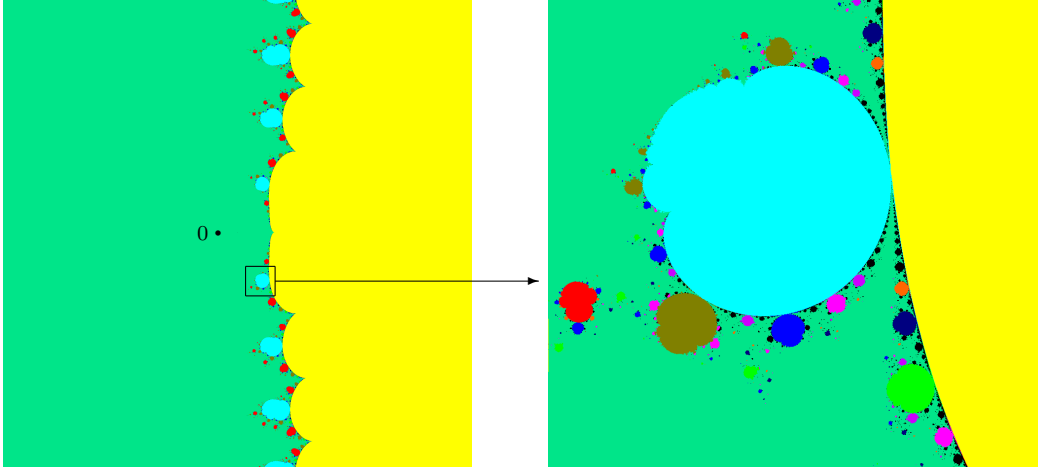


Figure 2: Left: The  $\lambda$ -plane of the meromorphic family  $f_\lambda$ , showing the dynamics of the free asymptotic value  $\lambda$ . Dimensions  $[-12, 12] \times [-12, 12]$ . Right: Zoom in on a shell component of period two (cyan-blue). Color coding is explained in the text. Dimensions  $[0.5, 1.9] \times [-3.25, 1.92]$

As in Figure 1, the color represents the period of the attracting orbit that the free asymptotic value  $\lambda$  is attracted to. It does not reflect the behavior of the orbit of the other asymptotic value  $-\mu$ . In the unbounded component on the left (green) both  $\lambda$  and  $-\mu$  are attracted to the origin. Observe that this is a capture component as opposed to a shell component; the origin is always an attracting fixed point so it attracts at least one of the asymptotic values and in this component it attracts both. In the unbounded yellow shell component on the right,  $\lambda$  is attracted to an attracting fixed point different from zero while  $-\mu$  is attracted to zero. In the cyan-blue bounded shell components,  $-\lambda$  is attracted to a cycle of period 2; in the red ones the attracting cycle has period three, etc. Bifurcations occur at parabolic parameters as usual, giving rise to shell components of all periods attached to the boundary of any given shell component.

In Figure 3 we again plot the  $\lambda$ -plane but we show the dynamics of the second asymptotic value  $-\mu$ . On the unbounded green capture component both asymptotic values are attracted to the origin. On the complement,  $\lambda$  is attracted to the origin and  $-\mu$  is free. The yellow region is a shell component of period one which should have a virtual center at the parameter  $\lambda = 1/3$ , (where  $\mu = \infty$ ) but this is a parameter singularity.

Observe that  $\mu$  is not an affine function of the parameter  $\lambda$ ; this explains why this period one component is bounded. In fact, if we were to reparametrize the family so that  $\mu$  depended affinely on the parameter, we would see the same picture as in Figure 3 because of the symmetry in the equation connecting  $\mu$  and  $\lambda$ .

(c) Figure 4 shows the parameter plane of the tangent family<sup>1</sup>  $T_\lambda(z) = \lambda \tan z$ . As we

<sup>1</sup> Note that the maps  $\lambda \tan z$  and  $\lambda \tanh z$  are conjugate under  $z \rightarrow iz$  so they are dynamically the same.

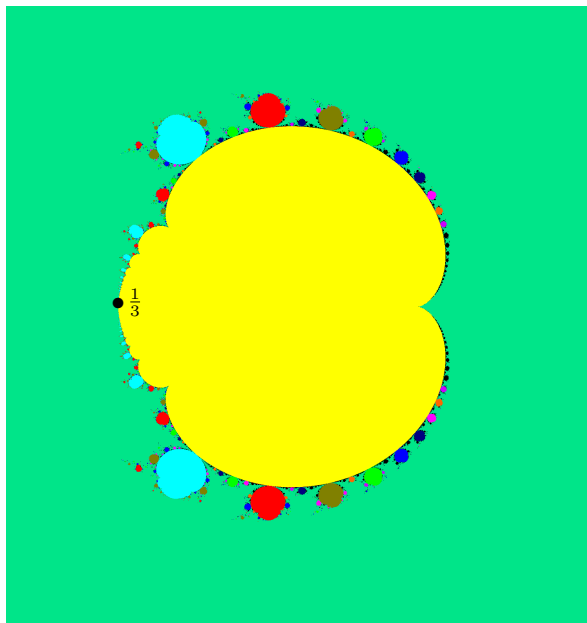


Figure 3: The  $\lambda$ - plane of the family  $f_\lambda$ , showing the dynamics of the second asymptotic value  $-\mu = \lambda/(1 - 3\lambda)$ . Dimensions  $[0.28, 0.44] \times [-0.08, 0.08]$ .

saw in Section 4.1, this is a dynamically natural slice of  $\mathcal{F}_2$  in the relaxed sense: the asymptotic values  $\lambda_1 = \lambda i, \lambda_2 = -\lambda i$  are tied together by the relation  $\lambda_1 = -\lambda_2$ . There is a lot of symmetry in this family. We have

$$T_\lambda(-z) = T_{-\lambda}(z) \text{ and } T_\lambda(-z) = -T_\lambda(z) \text{ so that}$$

$$T_{-\lambda}(-z) = T_\lambda(z).$$

In addition

$$T_{\bar{\lambda}}(z) = \overline{T_\lambda(\bar{z})}.$$

This says, for example, that if for some  $\lambda_0$ ,  $z_1$  is a fixed point of  $T_{\lambda_0}$ , then  $T_{-\lambda_0}(z_1) = -z_1$  so that  $z_1$  is a period 2 point of  $T_{-\lambda_0}$ . It also says that complex conjugate values of  $\lambda$  have conjugate periodic cycles.

The figure shows the  $\lambda$  plane where we can observe these symmetries. We only follow the orbit of ONE asymptotic value,  $\lambda i$ , and we color the components based on the period of the cycle it is attracted to. The colors do not reflect the behavior of the orbit of the other asymptotic value. Thus, we see the same color for a  $\lambda$  value for which there are two separate attracting periodic cycles of period  $2k$  as we do when there is a single attracting periodic cycle of period  $2k$  attracting both asymptotic values.

There is a single non-simply connected component, the punctured unit disk, for which both asymptotic values are attracted to the origin. There are two unbounded components, one on the right (yellow) and one on the left (cyan-blue). In the one on the right,  $\lambda i$  is attracted to a fixed point  $a_0$  with multiplier  $\rho_0$ , and  $-\lambda i$  is attracted to  $-a_0$  with the same multiplier. It has the same color as the unit disk. In the unbounded

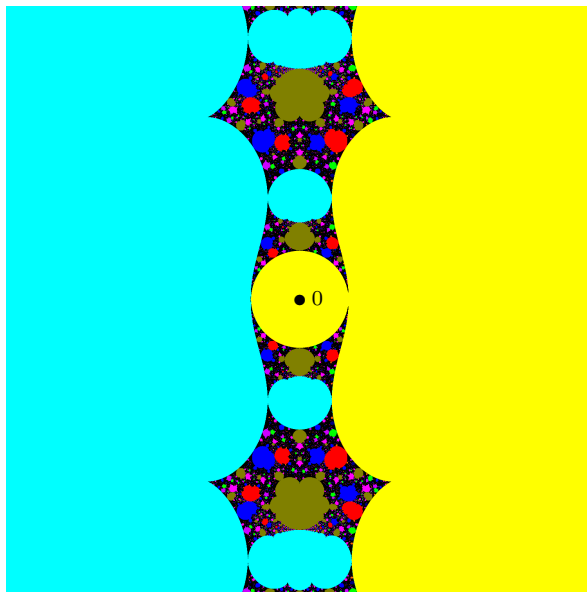


Figure 4: The parameter plane of the family  $T_\lambda(z) = \lambda \tan z$ . Dimensions  $[-6, 6] \times [-6, 6]$ .

component on the left, both  $\lambda i$  and  $-\lambda i$  are attracted to the attracting period two cycle  $a_0, -a_0$  and its multiplier is  $\rho_0^2$ . Neither of these components has a finite virtual center. Although the left one is colored for period 2, it cannot have a finite virtual center because if  $\lambda^*$  were a finite virtual center,  $-\lambda^*$  would be a virtual center for the unbounded component on the right of period 1, contradicting Corollary 6.15.

Note that, except for the punctured disk, each bounded component is paired with another bounded component; the two have a common virtual center and are tangent there. The two unbounded components can be thought of as paired at infinity. The relationship between these component pairs is discussed in detail in [KK97].

- (d) Figure 5 is another dynamically natural slice of  $\mathcal{F}_2$  in the relaxed sense. In this example the origin is a fixed point again but its multiplier can vary. Instead of fixing this multiplier, we require that one asymptotic value is always at  $\pi i$  and that its first iterate lands on the fixed point at the origin. Functions of the form

$$f_\lambda(z) = \frac{e^z - e^{-z}}{(1/\lambda)e^z + (i/\pi)e^{-z}} \quad (7.1)$$

have this property. They have asymptotic values at  $\lambda$  and at  $\pi i$  and  $f_\lambda(\pi i) = f_\lambda(0) = 0$ . The multiplier at the fixed point is equal to the parameter  $\lambda$ . The parameter singularities are at 0, where the denominator is infinite, and at  $\pi i$ , where the two singular values collide and the denominator is zero.

In the figure, we show the dynamics of the free asymptotic value  $\lambda$ . The parameter  $\lambda = 0$  is a parameter singularity and belongs to the central yellow disk-like component. This component corresponds to parameter values for which the free asymptotic value

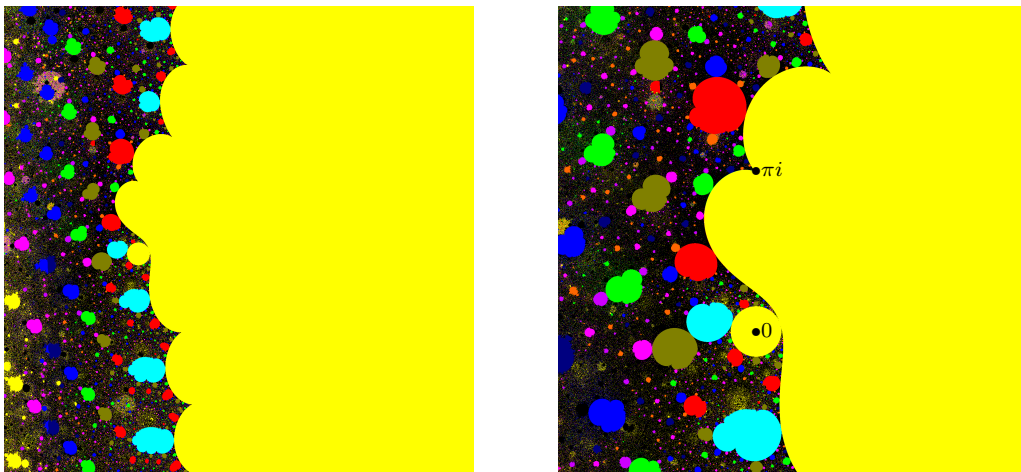


Figure 5: Left: The parameter plane of the family  $f_\lambda(z)$  in (7.1). Dimensions  $[-6, 15] \times [-10, 11]$ . Right: Zoom showing the two parameter singularities. Dimensions  $[-4, 5.5] \times [-3, 6.5]$

is attracted to the origin. It is conformally homeomorphic to  $\mathbb{D}^*$  and the multiplier is univalent. The puncture is located at  $\lambda = 0$ .

In the unbounded component on the right (yellow),  $\lambda$  is attracted to an attracting fixed point different from the origin. The second parameter singularity  $\lambda = -\pi i$ , is located on the boundary of this component. When  $\lambda$  approaches this value, the multiplier tends to  $-1$ . Since the parameter is singular, the usual period doubling bifurcation does not occur and we see a cusp instead of an attached period two component.

**Example 2.** In this set of examples, the functions are formed by pre and post-compositing a function in  $\mathcal{F}_2$  with a quadratic polynomial.

(a) Figure 6 shows the dynamically natural slice for the family

$$g_\lambda(z) = \frac{e^{z^2} - e^{-z^2}}{\frac{1}{\lambda}e^{z^2} + \frac{1}{\sqrt{\pi i}}e^{-z^2}}$$

formed by pre-composition of a function with asymptotic values at  $\lambda$  and  $-\sqrt{\pi i}$  with the quadratic polynomial  $Q(z) = z^2$ . Recall from Section 4.1 that  $g(z)$  has two double asymptotic values at  $\lambda$  and  $-\sqrt{\pi i}$  and a super attractive fixed point at the origin. The slice is natural because  $g_\lambda(-\sqrt{\pi i}) = 0$ . The figure represents the  $\lambda$ -plane. The green components are capture components where the orbit of  $\lambda$  falls into the basin of attraction of the origin. In the unbounded yellow shell components,  $\lambda$  is attracted to an attracting (not super attracting) fixed point. In the cyan-blue (bounded) components it is attracted to a cycle of period two, in the red ones to a cycle of period three and in the brownish green ones to a cycle of period four. The components come in pairs because the asymptotic values have multiplicity two. This is discussed in [?].

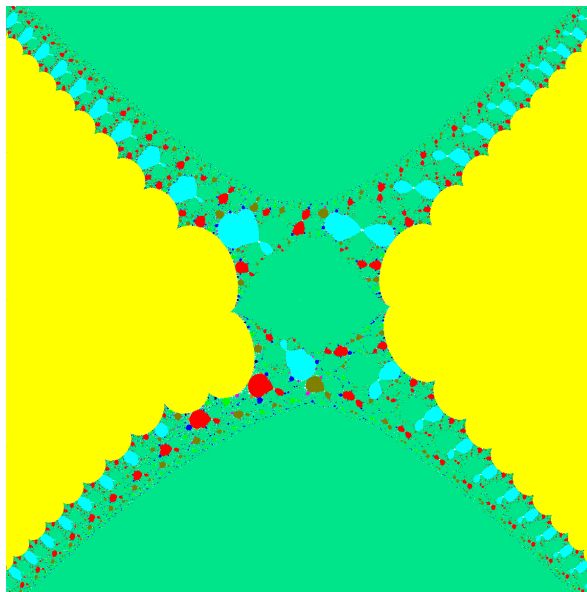


Figure 6: Dynamically natural slice for functions in  $\mathcal{F}_2$  pre-composed with a quadratic polynomial.

(b) In the left plot of Figure 7 we see the relaxed dynamically natural slice for the family

$$\lambda \tanh^2 z$$

discussed in Example (c) of Section 4.1. These functions are obtained by post-composition of a function in  $\mathcal{F}_2$  with a quadratic polynomial. To make the slice dynamically natural in the relaxed sense, we specify that the two asymptotic values coalesce, creating a single asymptotic value  $\lambda$  of multiplicity two; in addition, we make the origin a fixed critical point. The origin is the only parameter singularity for the family so  $\Lambda = \mathbb{C} \setminus \{0\}$ .

In the center of the figure we see a green capture component formed by parameters for which the asymptotic value belongs to the immediate basin of 0. This component is doubly connected because of the puncture at  $\lambda = 0$ . The remaining green components are capture components for higher iterates.

In the two yellow shell components the period is one. They are unbounded in accordance with Proposition 6.11. The remaining shell components are all of higher period and are all bounded. We see that they are grouped into quadruples that share a virtual center. This is because the asymptotic value has two asymptotic tracts and each asymptotic tract has two pre-asymptotic tracts. This example is investigated further in [CK].

Compare this figure to the right plot in Figure 7 which shows the parameter plane for the family  $\lambda \tanh z^2$ . It is essentially the “square root” of this slice, since both maps are semiconjugate by  $z^2$ . The asymptotic values are now double.

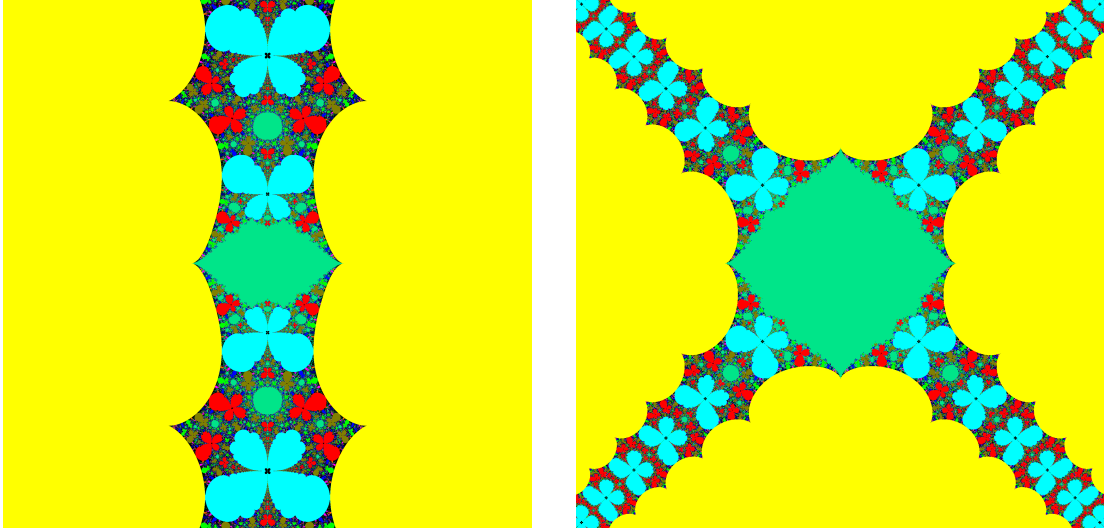


Figure 7: Left: Parameter plane of the family  $\lambda \tanh^2 z$ . Dimensions  $[-6, 6] \times [-6, 6]$ .  
Right: Parameter plane of the family  $\lambda \tanh z^2$ . Dimensions  $[-3, 3] \times [-3, 3]$

## 8 Appendix

Any meromorphic function that is a branched cover of the sphere of finite degree  $d$  is a rational function and so can be expressed as a quotient of relatively prime polynomials at least one of which has degree  $d$ . The  $2d + 1$  coefficients of these polynomials define a natural embedding of the space of such functions into  $\mathbb{C}^{2d+1}$ . Up to affine conjugation then, these functions are represented as a complex analytic manifold of dimension  $2d - 2$ . This family is denoted by  $Rat_d$  in the literature.

Not all holomorphic families of meromorphic functions have such an obvious representation as a complex manifold. In this appendix we consider some transcendental families that do. These are the families that most of our examples are drawn from. The functions have explicit expressions that involve complex constants. We show that under quasiconformal deformation, the deformed function has a similar expression in which only the constants are changed. Thus the constants determine the embedding of the manifold into  $\mathbb{C}^n$  for the appropriate  $n$ .

We begin with families of meromorphic functions with  $p < \infty$  asymptotic values, no critical points and a single essential singularity at infinity. We denote these by  $\mathcal{F}_p$ . (See [DK88, KK97, EG09] for further discussion.)

These functions have a particularly nice characterization. Recall that the Schwarzian derivative of a function  $g$  is defined by

$$S(g) = (g''/g')' - \frac{1}{2}(g''/g')^2. \quad (8.1)$$

Nevanlinna, [Nev70, Nev32], Chap X1, proved

**Theorem 8.1.** *Every meromorphic function  $g$  with  $p < \infty$  asymptotic values and  $q < \infty$  critical points has the property that its Schwarzian derivative is a rational function of degree*

$p+q-2$ . If  $q=0$ , the Schwarzian derivative is a polynomial  $P(z)$ . In the opposite direction, for every polynomial function  $P(z)$  of degree  $p-2$ , the solution to the Schwarzian differential equation  $S(g) = P(z)$  is a meromorphic function with exactly  $p$  asymptotic values and no critical points.

We sketch the proof of the theorem here and refer the interested reader to the literature. The proof follows from the construction of a function with  $p$  asymptotic values and no critical points as a limit of rational functions with  $p$  branch points. Letting the order of the branching at some or all of these  $p$  points increase, one obtains a sequence of rational functions. In the limit, the images of the branch points whose order goes to infinity become logarithmic singularities and the limit function has finitely many branch points and finitely many logarithmic singularities. The limit function is a parabolic covering map from the plane to itself. The Schwarzian derivatives of the rational functions in the sequence are again rational functions with degree determined only by the number of branch points, not their order. The limit of the Schwarzian derivatives is the Schwarzian derivative of the limit and so must be rational.

If all the branch points become asymptotic values in the limit, the limit function has no critical points or critical values and hence its derivative never vanishes. It follows from the definition of the Schwarzian that in the limit, it must be a polynomial.

It is classical, ( see e.g. [Hil76]), that solutions to the Schwarzian equation  $S(g) = P(z)$  are related to solutions of the linear second degree ordinary differential equation

$$w'' + \frac{1}{2}P(z)w = 0. \quad (8.2)$$

Such an equation has a two dimensional space of holomorphic solutions. If  $w_1, w_2$  are a pair of linearly independent solutions, and

$$g = \frac{aw_2 + bw_1}{cw_2 + dw_1}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1,$$

it is easy to check that  $S(g) = P(z)$ . Moreover, if  $g$  is any solution of the Schwarzian equation,  $w = \sqrt{1/g'}$  is a solution of the linear equation.

*Remark 8.2.* One of the basic features of the Schwarzian derivative is that it satisfies the following cocycle relation: if  $f, g$  are meromorphic functions then

$$S(g \circ f(z)) = S(g(f))f'(z)^2 + S(f(z)). \quad (8.3)$$

In particular, if  $T$  is a Möbius transformation,  $S(T(z)) = 0$  and  $S(T \circ g(z)) = S(g(z))$  so that post-composing by  $T$  does not change the Schwarzian. Under pre-composition by a Möbius transformation the Schwarzian behaves like a quadratic differential. In particular, pre-composing by an affine transformation multiplies the Schwarzian by a constant.

This means that if we find a specific pair  $(w_1^*, w_2^*)$  of solutions to the second order linear equation and set  $g^* = w_2^*/w_1^*$ , then every solution of the Schwarzian equation  $S(g) = P(z)$  has the formula

$$g = \frac{ag^* + b}{cg^* + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1,$$



In particular, if  $p = 2$ ,  $P(z)$  is identically constant. Using one of the constants in an affine conjugation, we may assume  $P \equiv -1/2$ ; then a specific pair of solutions to equation 8.2 is  $w_1^* = e^{-\frac{z}{2}}$ ,  $w_2^* = e^{\frac{z}{2}}$  so we have  $g^* = w_2^*/w_1^* = e^z$ . The functions in the family  $\mathcal{F}_2$  have the form

$$g(z) = \frac{ae^z + b}{ce^z + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1.$$

Since there is one more degree of freedom from the affine conjugation, we see this is a two dimensional family. We have discussed several dynamically natural slices of  $\mathcal{F}_2$  in this paper.

For  $\mathcal{F}_3$ ,  $P(z)$  is linear and two specific solutions to the second order linear equation

$$w'' + \frac{1}{2}\zeta w = 0$$

are given by the Airy functions<sup>2</sup>  $Ai(\zeta), Bi(\zeta)$ . Setting  $g^*(\zeta) = Ai(\zeta)/Bi(\zeta)$  we obtain a solution with three asymptotic values,  $0, i$  and  $-i$ . Since we are interested in the dynamics, we may conjugate by the affine transformation  $\zeta = rz + s$ . We may thus transform any function whose Schwarzian is linear, and hence any function in  $\mathcal{F}_3$ , up to affine conjugation, to one given by the formula

$$g(\zeta) = \frac{ag^*(\zeta) + b}{cg^*(\zeta) + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1.$$

The asymptotic values of  $g$  are  $\frac{b}{d}, \frac{ai+b}{ci+d}, \frac{-ai+b}{-ci+d}$ .

We could define a dynamically natural slice, for example, by choosing the coefficients  $a, b$  and  $d$  so that

$$g(1) = 1, \text{ and } g'(1) = 1/2.$$

It is not hard to compute that the remaining coefficient  $c$  is an affine function of the asymptotic value  $\frac{b}{d}$ .

There are standard functions that solve the second order equation when the coefficient polynomial is of degree two or three, and they give formulas for functions in  $\mathcal{F}_4$  and  $\mathcal{F}_5$ .

## Compositions

The following theorem shows us how we can find more families of functions with singular points which are closed under quasiconformal deformation. We have

**Theorem 8.3.** *Let  $f_0 \in \mathcal{F}_p$*

1. *Suppose  $g_0 = f_0 \circ Q_0$  is a function such that  $Q_0$  is a polynomial of degree  $d$  and suppose that  $g$  is a meromorphic function quasiconformally conjugate to  $g_0$ . Then  $g$  can be expressed as  $g = f \circ Q$  for some function  $f \in \mathcal{F}_p$  and some polynomial  $Q$  of degree  $d$ .*

---

<sup>2</sup> These are named after a physicist G.B.Airy. Others have studied this equation and solutions are expressed in terms of Bessel functions and Gamma functions<sup>3</sup> We won't write the formulas but will describe the properties we need.

2. Suppose  $R_0 \in \text{Rat}_d$  is rational of degree  $d$  and  $h_0 = R_0 \circ f_0$ . Then if  $h(z)$  is quasiconformally conjugate to  $h_0$ ,  $h$  can be expressed as  $h = R \circ f$  for functions  $R \in \text{Rat}_d$  and  $f \in \mathcal{F}_p$ .

*Remark 8.4.* The proof of part 1 of the theorem works if  $Q_0$  is replaced by a rational function  $R_0 \in \text{Rat}_d$  but then the composed function  $g_0$  has essential singularities at the poles of  $R_0$ .

*Remark 8.5.* Note that in part 1,  $g = f \circ Q$  is a function with rational Schwarzian. For each asymptotic value of  $f$  of multiplicity  $m$ ,  $g$  has an asymptotic value of multiplicity  $dm$ ; moreover  $g$  has the same critical points as  $Q$ , namely  $2d - 2$  critical points counted with multiplicity,  $d - 1$  of which are at infinity. In part 2, however, it is no longer true that the Schwarzian of the composed function  $h = R \circ f$  is a rational function. For example, it may have infinitely many critical points.

*Remark 8.6.* Observe that if  $f \in \mathcal{F}_p$  and  $Q$  is a degree  $d$  polynomial then  $Q \circ f$  and  $f \circ Q$  are semiconjugate by a degree  $d$  polynomial – in fact by  $Q$ .

*Proof.* The proof of both parts of the theorem is essentially the same. We therefore carry it out only for part 1.

In part 1, let  $\phi^\mu$  be a quasiconformal homeomorphism with Beltrami coefficient  $\mu$  such that

$$g_\mu = \phi^\mu \circ g_0 \circ (\phi^\mu)^{-1}$$

is meromorphic. We can use  $f_0$  to pull back the complex structure defined by  $\mu := \bar{\partial}\phi^\mu / \partial\phi^\mu$  to obtain a complex structure  $\nu = f_0^*\mu$  such that the map

$$f_\mu = \phi^\mu \circ f_0 \circ (\phi^\nu)^{-1}$$

is meromorphic. Note that this is not a conjugacy since it involves two different homeomorphisms.

We can now write

$$g_\mu = \phi^\mu \circ f_0 \circ (\phi^\nu)^{-1} \circ \phi^\nu \circ Q_0 \circ (\phi^\mu)^{-1}$$

and set

$$Q_\mu = \phi^\nu \circ Q_0 \circ (\phi^\mu)^{-1}.$$

Again this is not a conjugacy but it is meromorphic since  $g_\mu$  and  $f_\mu$  are homeomorphisms.

The main point here is that although  $f_\mu$  is not a conjugate of  $f_0$ , since the quasiconformal maps  $\phi^\mu$  and  $\phi^\nu$  are homeomorphisms, the map  $f_\mu$  is a meromorphic map with the same topology as  $f_0$ ; that is, it has  $p$  asymptotic values and no critical values. By Nevanlinna's theorem, Theorem 8.1,  $f_\mu$  belongs to  $\mathcal{F}_p$ . Similarly, although  $Q_\mu$  is not defined as a conjugate of  $Q_0$ , since the quasiconformal maps  $\phi^\nu$  and  $\phi^\eta$  are homeomorphisms, the map  $Q_\mu$  is a meromorphic map with the same topology as  $Q_0$ ; that is, it is a degree  $d$  branched covering of the Riemann sphere with the same number of critical points and the same branching as  $Q_0$  and thus it must be a polynomial of degree  $d$ .  $\square$

Other families of functions have been shown to be invariant under quasiconformal deformation respecting the dynamics. We state the theorems here and refer the reader to the cited literature for proof.

**Theorem 8.7.** *Functions of type  $Pe^Q$  for  $P, Q$  polynomials are, up to affine conjugacy, invariant under quasiconformal deformation respecting the dynamics.*

This is the content of [CJK] Theorem 4; the proof uses an analysis of the topological mapping properties of these maps as described in detail in [Zak10].

Another example of this phenomenon is proved in [BF10] and in [Den13].

**Theorem 8.8.** *Any entire transcendental map of finite order with two singular values, one of which is a fixed simple critical point, and the other an asymptotic value with one finite preimage, is affine conjugate to*

$$D_a(z) = a(e^z(z+1) + 1), \quad a \in \mathbb{C}^*$$

for some  $a \in \mathbb{C}$ . The affine conjugacy is used to make the parameter unique by setting the critical point at 0 and the asymptotic value at  $a$ .

## References

- [BE95] W. Bergweiler and A. Eremenko, *On the singularities of the inverse of a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 355–373.
- [Ber93] Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188.
- [BF10] R. Berenguel and N. Fagella, *An entire transcendental family with a persistent siegel disc*, J. Diff. Eq. and App. **16** (2010), 523–553.
- [BF14] Bodil Branner and Núria Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, vol. 141, Cambridge University Press, Cambridge, 2014, With contributions by Xavier Buff, Shaun Bullett, Adam L. Epstein, Peter Haïssinsky, Christian Henriksen, Carsten L. Petersen, Kevin M. Pilgrim, Tan Lei and Michael Yampolsky. MR 3445628
- [BKL91] I. N. Baker, J. Kotus, and Yi Nian Lü, *Iterates of meromorphic functions. III. Preperiodic domains*, Ergodic Theory Dynam. Systems **11** (1991), no. 4, 603–618.
- [BKY91] I. N. Baker, J. Kotus, and Lü Yinian, *Iterates of meromorphic functions. I*, Ergodic Theory Dynam. Systems **11** (1991), no. 2, 241–248.
- [BKY92] I. N. Baker, J. Kotus, and Lü Yinian, *Iterates of meromorphic functions. IV. Critically finite functions*, Results Math. **22** (1992), no. 3-4, 651–656. MR 1189754
- [Bol99] A. Bolsch, *Periodic Fatou components of meromorphic functions*, Bull. London Math. Soc. **31** (1999), no. 5, 543–555. MR 1703869 (2000e:30046)
- [BR75] I. N. Baker and P. Rippon, *Iteration of exponential functions*, Ann. Acad. Sci. Fenn. Math. Ser. A **1** (1975), no. 2, 277–283.

- [CG93] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 1230383 (94h:30033)
- [CJK] T. Chen, Y. Jiang, and Linda Keen, *Bounded geometry and characterization of some transcendental maps*, Indiana J. of Math.
- [CK] T. Chen and L. Keen, *The dynamics of the families  $\lambda \tanh^n z$* , Preprint.
- [Den13] A. Deniz, *Entire transcendental maps with two singular values*, Ph.D. thesis, Roskilde University / Universitat de Barcelona, 2013.
- [DFJ02] R. L. Devaney, N. Fagella, and X. Jarque, *Hyperbolic components of the complex exponential family*, Fundamenta Mathematicae **174** (2002), 193–215.
- [DK88] R. L. Devaney and L. Keen, *Dynamics of tangent*, Dynamical Systems, Proceedings, University of Maryland, Springer-Verlag Lecture Notes in Mathematics. **1342** (1988), 105–111.
- [EG09] Alexandre Eremenko and Andrei Gabrielov, *Analytic continuation of eigenvalues of a quartic oscillator*, Comm. Math. Phys. **287** (2009), no. 2, 431–457. MR 2481745 (2010b:34189)
- [EL92] A. È. Èrëmenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.
- [Gau12] Thomas Gauthier, *Strong bifurcation loci of full Hausdorff dimension*, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012), no. 6, 947–984 (2013). MR 3075109
- [GK86] Lisa R. Goldberg and Linda Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynam. Systems **6** (1986), no. 2, 183–192.
- [GK08] Piotr Galazka and Janina Kotus, *The straightening theorem for tangent-like maps*, Pacific J. Math. **237** (2008), no. 1, 77–85. MR 2415208
- [Hei57] Maurice Heins, *Asymptotic spots of entire and meromorphic functions*, Ann. of Math. (2) **66** (1957), 430–439. MR 0094457 (20 #975)
- [Her98] M. E. Herring, *Mapping properties of Fatou components*, Ann. Acad. Sci. Fenn. Math. **23** (1998), no. 2, 263–274. MR 1642181 (2000b:30032)
- [Hil76] Einar Hille, *Ordinary differential equations in the complex domain*, Wiley-Interscience, New York-London-Sydney, 1976, Pure and Applied Mathematics. MR 0499382
- [Ive06] F. Iversen, *Recherches syr les fonctions inverses*, Comptes Rendus **143** (1906), 877–879.
- [KK97] L. Keen and J. Kotus, *Dynamics of the family  $\lambda \tan z$* , Conformal geometry and dynamics **1** (1997), 28–57.

- [KY06] Linda Keen and Shenglan Yuan, *Parabolic perturbation of the family  $\lambda \tan z$* , Complex dynamics, Contemp. Math., vol. 396, Amer. Math. Soc., Providence, RI, 2006, pp. 115–128. MR 2209090
- [McM94] Curtis T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994. MR 1312365 (96b:58097)
- [Nev32] Rolf Nevanlinna, *Über Riemannsche Flächen mit endlich vielen Windungspunkten*, Acta Math. **58** (1932), no. 1, 295–373. MR 1555350
- [Nev70] ———, *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer-Verlag, New York-Berlin, 1970. MR 0279280 (43 #5003)
- [Pom92] Christian Pommerenke, *Boundary behaviour of conformal maps*, Springer Verlag, 1992.
- [RG03] Lasse Rempe-Guillen, *Dynamics of exponential maps*, Ph.D. thesis, Christian-Albrechts-Universität Kiel, 2003.
- [Sch03] D. Schleicher, *Attracting dynamics of exponential maps*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 3–34.
- [Sul85] Dennis Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), no. 3, 401–418.
- [tt60] Lars AHLFORS and Lipman BERS, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.
- [Zak10] Saeed Zakeri, *On Siegel disks of a class of entire maps*, Duke Math. J. **152** (2010), no. 3, 481–532. MR 2654221
- [Zhe10] Jianhua Zheng, *Value distribution of meromorphic functions*, Tsinghua University Press, Beijing; Springer, Heidelberg, 2010. MR 2757285 (2012d:30002)